AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

VOLUME LXIII

1941

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume LXIII, Number 1
JANUARY, 1941

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

CONTENTS

The American Journal of Mathematics will appear four times yearly.

The subscription price of the Journal for the current volume is \$7.50 (foreign postage 50 cents); single numbers \$2.00.

A few complete sets of the Journal remain on sale.

Papers intended for publication in the Journal may be sent to any of the Editors.

Editorial communications may be sent to Professor F. D. Murnaghan at The Johns Hopkins University.

Subscriptions to the Journal and all business communications should be sent to The Johns Hopkins Press, Baltimore, Maryland, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918.

PRINTED IN THE UNITED STATES OF AMERICA
BY J. H. FURST COMPANY, BALTIMORE, MARYLAND

F

se
u
th
su
ex
or
by
as
a
th
sp
m
so
W
in
pe
sp
fo

pl
A
Le
tie
th
bu
A
th
W

ha
as

FIXED-POINT THEOREMS FOR PERIODIC TRANSFORMATIONS.*

By P. A. Smith.

The simplest fixed-point theorems for transformations of finite period seem to be those which assert that fixed points must exist if the space M under transformation is simply connected in some specified sense. In the first theorem of this sort, obtained by the author [2] in 1934, M was taken to be a subset of euclidean n-space in which all singular spheres of dimensions not exceeding $pn - n - 1$ could be contracted to points; p here denotes the period of the transformation and is assumed to be prime.[1] It has recently been shown by S. Eilenberg [2] that these assumptions of homotopy could be replaced by assumptions of homology; it is still required, however, that M be a subset of a euclidean space and that p be a prime. We propose now to contribute to these results first, by allowing spaces which are not immersible in euclidean spaces, and secondly, by allowing periods which are powers of primes. Our methods of proof are quite different from the earlier ones and, we believe, somewhat simpler—at least as concerns transformations of prime period. With regard to transformations of perfectly arbitrary periods it remains an interesting open question whether or not they must, in general, admit fixed points assuming even that the space under transformation is a euclidean n-space. We have been able to answer in the affirmative only when $n \leqq 3$ and, for suitably regular transformations, when $n = 4$ (sections 5, 6).

1. Preliminaries. Let K be a finite simplicial complex which is simplicially transformed into itself by a homeomorphism T of period p, p a prime. Assume T to be such that the invariant simplexes form a sub-complex K^0. Let $\mathfrak{g}$ be a coefficient group for chains and homologies in K. The transformation T induces a transformation of the chains of K which is permutable with the boundary operator Δ. This is perfectly clear for chains of dimension > 0, but a word should perhaps be said concerning 0-chains. Let E be a vertex. Associated with E are two oriented 0-simplexes, namely $+E, -E$ (ordinarily the $+$ is not written). Similarly, with TE are associated $+(TE), -(TE)$. We shall agree that T transforms oriented 0-simplexes according to the rule

* Received April 2, 1940.

[1] Although the hypothesis that p be prime was not specifically stated in [2], we have been aware for some time that the argument is not valid for composite p. The assertion in the eighth line of section 1 holds only if p is a prime.

[2] Duke Journal 6 (1940), pp. 428-437.

$T(\epsilon E) = \epsilon\,(TE)$ where $\epsilon = +$ or $-$. It is easy to see that with this convention Δ, T are permutable without exception. With regard to cycles, we shall agree that a 0-chain is a 0-cycle if and only if the sum of its coefficients is zero. Thus $E - TE$ is a cycle but $\pm E$ is not.

Let

$$\sigma = 1 + T + \cdot \cdot \cdot + T^{p-1}, \qquad \delta = 1 - T.$$

We shall use the symbol ρ to stand for σ or δ. Having agreed in a given discussion which of these operators is to be ρ, the other will be denoted by $\bar{\rho}$. We shall say that a chain X is of type ρ if it is expressible in the form ρY. The null chain is to be regarded as of both possible types. *In order that a chain X in $K - K^0$ be of type ρ it is necessary and sufficient that $\bar{\rho}X = 0$.* The necessity of the condition follows from the obvious relations $\rho\bar{\rho} = \bar{\rho}\rho = 0$. The proof of sufficiency is less immediate but perfectly straightforward (see [3], p. 142).

Suppose C is a cycle of type ρ. If, modulo K^0, C is the boundary of a chain of the same type, we shall write $C \simeq 0 \bmod K^0$. Suppose that C_h, C_{h-1} are cycles of types ρ and $\bar{\rho}$ respectively. If there exists a chain X_h such that

$$C_h = \rho X_h, \qquad C_{h-1} = \Delta X_h,$$

we shall write $C_h : C_{h-1}$.

2. Let it now be understood that the coefficient group $\mathfrak{g}$ is $\mathfrak{p}$, the group of integers reduced mod p. Then if Y^0 is a chain in K^0 we have $\rho Y^0 = 0$; for, a simplex appearing in Y^0 with the coefficient x will appear in δY^0 with the coefficient 0 and in σY^0 with the coefficient px ($= 0 \bmod p$). It follows from this that every chain of type ρ lies in $K - K^0$. For, a chain Y can always be expressed in the form $Y' + Y^0$ where $Y^0 \subset K^0$, $Y' \subset K - K^0$. Then $\rho Y = \rho Y' \subset K - K^0$.

LEMMA 1. *Let C_h, C_{h-1} be cycles of types ρ, $\bar{\rho}$ such that $C_h : C_{h-1}$. If $C_h \simeq 0 \bmod K^0$, then also $C_{h-1} \simeq 0$ mod K^0.*

Proof. Assuming that $C_h \simeq 0 \bmod K^0$, there exist chains X_{h+1}, X_h, X^0 such that

(1) $\qquad C_h = \rho X_h, \quad C_{h-1} = \Delta X_h, \quad \Delta X_{h+1} = C_h + X^0, \quad X^0 \subset K^0.$

We may write

(2) $\qquad \Delta X_{h+1} = X_h + Z + Z^0 \; (Z^0 \subset K^0,\; Z \subset K - K^0).$

Then $\rho\Delta X_{h+1} = C_h + \rho Z + \rho Z^0$. Since $\rho Z^0 = 0$, it follows from the third

relation in (1) that $\rho Z = X^0$. But since $\rho Z \subset K - K^0$, we have $\rho Z = 0$, so that Z is of type $\bar{\rho}$. Hence if we operate on both sides of (2) with Δ we find that $C_{h-1} \simeq 0 \bmod K^0$.

LEMMA 2. *Let E be a vertex of K. If $K^0 = 0$, the cycle $\delta E = E - TE$ cannot be $\simeq 0$.*

Proof. Suppose on the contrary that there exists a chain X such that $\Delta\delta X = \delta E$. Let

$$Z = \Delta X - E. \tag{1}$$

Then $\delta Z = 0$ and therefore, since $K^0 = 0$, Z is of type σ, say $Z = \sigma W$. Consequently the sum of the coefficients of Z is zero (modulo p) and Z is a cycle. Since ΔX is a cycle it follows from (1) that $-E$ is a cycle, which is impossible (see section 1).

3. The fixed-point theorem. A *space* from now on will mean a Hausdorff space. Let M be a locally bicompact space. We shall say that M is *acyclic* mod p if for every bicompact set A there exists a bicompact set $A' \supset A$ such that relative to the coefficient group $\mathfrak{p}$, cycles in A are ~ 0 in A'. Cycles and homologies are to be understood in the sense of Čech [1].

THEOREM I(α). *Let p be a prime and M a finite dimensional locally bicompact space which is acyclic mod p. Every homeomorphic transformation of period p^α ($\alpha > 0$) of M into itself admits at least one fixed point.*

Proof of theorem I(1). Let T be a transformation of period p operating in M. Finite dimensionality means that there exists an n such that every covering of M (i. e. finite covering by open sets) has a refinement whose nerve is of dimension $\leqq n$. Observe that if B is a bicompact set, σB is an *invariant* bicompact set containing B. This makes it clear that there can be chosen $pn + n + 1$ invariant bicompact sets $A_0, A_1, \cdots$ such that

$$0 \neq A_0 \subset A_1 \subset \cdots \subset A_m \qquad (m = pn + p)$$

and such that cycles in A_i are ~ 0 in A_{i+1}. Let $N = \bar{A}_m$. We shall regard N as a bicompact subspace of M. T induces a transformation of period p of N into itself and it will be sufficient to show that this transformation (to be denoted also by T) admits a fixed point. It is easy to see that in the topology of N, as in that of M, cycles in A_i are ~ 0 in A_{i+1} ($i < m$).

Assume now that T admits no fixed point in N. Let us call a covering $\mathfrak{U}$ of N *regular* if (1) its vertices (i. e. component sets) are permuted among

themselves by T; (2) no vertex meets any of its images under $T, T^2, \cdots, T^{p-1}$; (3) the dimension of the complex (i. e. nerve of) $\mathfrak{U}$ is less than m. As a result of the second condition, the simplicial transformation induced by T in the complex $\mathfrak{U}$ admits no invariant simplex. We assert that there exist "arbitrarily fine" regular coverings. This is true because (a) like M, N admits arbitrarily fine coverings of dimension $\leqq n$; (b) if $\mathfrak{V}$ is a covering of N of dimension $\leqq n$, the covering $\mathfrak{U}$ obtained by superimposing $T\mathfrak{V}, \cdots, T^{p-1}\mathfrak{V}$ on $\mathfrak{V}$ is of dimension $\leqq pn + p - 1 < m$ and satisfies (1); since N is bicompact, $\mathfrak{U}$ can be made arbitrarily fine by taking $\mathfrak{V}$ sufficiently fine; (c) since points in N can be separated by open sets, $\mathfrak{U}$ can be made to satisfy (2) by taking $\mathfrak{V}$ sufficiently fine. Thus the totality $\mathfrak{U}$ of regular coverings is a complete system, and can serve for defining homology relations in N.

Consider a definite regular covering $\mathfrak{U}$. If h, k are given integers, there exists a regular refinement $\mathfrak{U}'$ of $\mathfrak{U}$ such that the $\mathfrak{U}$-cycle obtained by projecting into $\mathfrak{U}$ an h-dimensional $\mathfrak{U}'$-cycle in A_k will be the $\mathfrak{U}$-coordinate of a complete cycle in A_k and will therefore be ~ 0 in A_{k+1} (see [1]). Consequently, if $\mathfrak{U}_m$ is an arbitrarily chosen regular covering, there exist regular coverings $\mathfrak{U}_{m-1}, \cdots, \mathfrak{U}_0$ such that $\mathfrak{U}_m \supset \mathfrak{U}_{m-1} \supset \cdots \supset \mathfrak{U}_0$ and such that if π_i is a projection $\mathfrak{U}_i \rightarrow \mathfrak{U}_{i+1}$ $(i < m)$ and C_i an i-dimensional $\mathfrak{U}_i$-cycle in A_i, then $\pi_i C_i \sim 0$ in A_{i+1}. The particular choice of π_i is immaterial; let π_i be chosen in such a way that it will be permutable with T, hence with ρ and $\bar{\rho}$. It is easy to see that such "invariant" projections exist (see [3], p. 138).

Let $\rho_0, \rho_1, \cdots$ stand alternately for δ and σ starting with $\rho_0 = \delta$. Let X_0 be a $\mathfrak{U}_0$-vertex in A_0. Since A_0 is invariant, TX_0 is also in A_0. Hence $\rho_0 X_0$ is a 0-dimensional cycle in A_0 and therefore $\pi_0\rho_0 X_0 \sim 0$ in A_1, say $\pi_0\rho_0 X_0 = \Delta X_1$, $X_1 \subset A_1$. Then $\rho_1 X_1$ is a cycle because

$$\Delta\rho_1 X_1 = \rho_1 \Delta X_1 = \rho_1\rho_0\pi_1 X_0 = 0.$$

Since $\rho_1 X_1 \subset A_1$, we have $\pi_1\rho_1 X_1 \sim 0$ in A_2, say $\pi_1\rho_1 X_1 = \Delta X_2$. Continuing in this manner we obtain chains $X_0, \cdots, X_m$ such that

$$\Delta X_{i+1} = \pi_i \rho_i X_i \qquad (i = 0, \cdots, m-1; X_i = X_i(\mathfrak{U}_i)). \tag{1}$$

Let π_m be the identical projection of $\mathfrak{U}_m$ into itself and let

$$C_i = \rho_i(\pi_m \pi_{m-1} \cdots \pi_i) X_i \qquad (i = 0, \cdots, m).$$

Then $C_m, C_{m-1}, \cdots$ are $\mathfrak{U}_m$-cycles of types $\rho_m, \rho_{m-1}, \cdots$ respectively and as a consequence of (1),

$$C_m : C_{m-1} : \cdots : C_0.$$

The cycle C_0 is of the form δE where E is a U_m-vertex. Since dim $\mathfrak{U}_m < m$, we have $C_m = 0 \simeq 0$ and therefore by lemma 1 (with $K^0 = 0$) we have $C_0 = \delta E \simeq 0$ which, by lemma 2, is impossible.

4. Before proving theorem I(α) with $\alpha > 1$ it will be necessary to examine more closely the nature of the fixed-point set in the case $\alpha = 1$.

THEOREM II. *The totality L of fixed points which theorem I(1) asserts to be non-empty, is acyclic modulo p.*

Proof. Let B be a bicompact set containing points of L. It will be sufficient to prove that there exists a bicompact set $B' \supseteq B$ such that cycles in BL are ~ 0 in $B'L$. Consider the sets $A_0, \cdots, A_m$ in the proof of I(1). Obviously A_0 could have been *any* non-empty invariant bicompact set and therefore we may now suppose that $A_0 = \sigma B$. Let B' be a bounded open set containing $\bar{A}_m$. We shall show that for B' we may take the set $\bar{A}_m$.

Let V be a covering of the space N $(= \bar{A}_m)$ with dim $\mathfrak{V} \leqq n$, and let $\mathfrak{U}$, as above, be the union of $\mathfrak{V}$, $T\mathfrak{V}, \cdots, T^{p-1}\mathfrak{V}$. Then although dim $\mathfrak{U} \leqq pn + p = 1$, $\mathfrak{U}$ cannot be regular; it will in fact have at least one invariant simplex since N now contains fixed points. It is essential for our purposes that $\mathfrak{U}$ be modified in such a way that its invariant simplexes will lie in the fixed-point set L_N $(= LN)$. This modification can in fact be carried out. More precisely there exist coverings $\mathfrak{U}$ of N such that (1) the simplexes of $\mathfrak{U}$ are permuted among themselves by T; (2) the invariant simplexes of $\mathfrak{U}$ form a subcomplex $\mathfrak{U}^0$; the simplexes of $\mathfrak{U}^0$ and these only, are in L_N; (4) the simplexes of $\mathfrak{U} - \mathfrak{U}^0$ are of dimension $\leqq pn + p - 1$. The totality $\{\mathfrak{U}\}$ of these special coverings is a complete system.[3]

We now examine the homology relations in the space N and show that *every cycle in $L_N B$ is ~ 0 in L_N.* Since dim $L_N \leqq n$, we need only consider cycles of dimension $\leqq n$. Among the special coverings, let there be chosen $\mathfrak{U}_0, \cdots, \mathfrak{U}_m$ defined exactly as in the proof of I(1) and let π_i, ρ_i be also defined as before. Now let γ_{h-1} be a cycle in $L_N B$ with $h - 1 \leqq n$. Since $BL_N \subset A_{h-1}$, we have $\gamma \sim 0$ in A_h; in particular we may write

$$\gamma_{h-1}(\mathfrak{U}_h) = \Delta X_h \qquad (X_h = X_h(\mathfrak{U}_h) \subset A_h).$$

Since the simplexes of $\gamma(\mathfrak{U}_h)$, being in L_N, are in $\mathfrak{U}^0{}_h$, we have $\rho_h\ (\mathfrak{U}_h) = 0$. Therefore $\rho_h X_h$ is a cycle in A_h. Hence $\pi_h \rho_h X_h \sim 0$ in A_{h+1}. This is the first

[3] The detailed construction of the $\mathfrak{U}$'s will become clear from an examination of a similar construction described in detail in [3], pp. 132-138.

step in the building-up process described above which leads, in $\mathfrak{U}_m$, to the relations

$$C_m : C_{m-1} : \cdots : C_h \qquad (C_h = \pi_m \cdots \pi_i \rho_i X_i;\ i = h, \cdots, m).$$

Since C_m is of the form $\rho_m Z$, it is in $\mathfrak{U}_m - \mathfrak{U}^0{}_m$ (section 2). But since all simplexes of $\mathfrak{U}_m - \mathfrak{U}^0{}_m$ are of dimension $\leqq pn + n - 1 < m$, we have $C_m = 0 \simeq 0 \bmod \mathfrak{U}^0{}_m$ and hence by lemma 1, $C_h \simeq 0 \bmod \mathfrak{U}^0$. Let $\gamma' = \pi_m \cdots \pi_h \gamma$. From the definition of (Čech) cycle, $\gamma'(\mathfrak{U}_m) \sim \gamma(\mathfrak{U}_m)$ in BL_N. Let $\rho = \rho_h$, $X' = \pi_m \cdots \pi_h X$. Then

$$\rho X' \simeq 0 \bmod \mathfrak{U}^0{}_m;\ \Delta X' = \gamma'.$$

The first of these relations implies the existence of $\mathfrak{U}_m$-chains X^0, Y such that

$$\Delta \rho Y = \rho X' + X^0 \qquad (X^0 \subset \mathfrak{U}^0{}_m). \tag{1}$$

Let Z^0 be that subchain of $\Delta Y - X' - X^0$ which is in $U^0{}_m$ and Z the remainder. Then

$$\Delta Y = X' + X^0 + Z + Z^0 \qquad (Z^0 \subset \mathfrak{U}^0{}_m;\ Z \subset \mathfrak{U}_m - \mathfrak{U}^0{}_m). \tag{2}$$

If we operate on both sides of (2) by ρ and take into account the relation (1) and the fact that $\rho X^0 = \rho Z^0 = 0$, we find that $\rho Z = 0$. Hence by lemma 1 we may write $Z = \bar{\rho} W$. If we insert this into (2) and then operate on both sides of (2) by Δ, we have

$$0 = (\gamma' + \Delta X^0 + \Delta Z^0) + \Delta \bar{\rho} W.$$

The chain in parenthesis is in $\mathfrak{U}^0{}_m$, whereas $\Delta\bar{\rho} W$ is in $\mathfrak{U}_0 - \mathfrak{U}^0{}_m$. Consequently both chains are null and γ' is therefore the boundary of a chain in $\mathfrak{U}^0{}_m$. By the properties of special coverings, chains in $\mathfrak{U}^0{}_m$ are in L_N. Hence $\gamma(\mathfrak{U}_m) \sim \gamma'(\mathfrak{U}_m) \sim 0$ in L_N. Since $\mathfrak{U}_m$ is an arbitrary special covering, we have $\{\gamma(\mathfrak{U}_m)\} = \gamma \sim 0$ in L_N.

We now return to the space M. Let Γ be a cycle in BL. Then since $B \subset A_m$, Γ may evidently be regarded as being identical with a cycle which belongs to the space N. This cycle is in BL_N and is therefore homologous to zero in L_N. It is not difficult to see, then, that $\Gamma \sim 0$ in NL, and therefore, since $N = \bar{A}_m = B'$, $\Gamma \sim 0$ in $B'L$, which completes the proof.

5. *Proof of Theorem $I(\alpha)$ with $\alpha > 1$.* Assume the truth of $\mathrm{I}(\beta)$ for $\beta < \alpha$. Let T be a transformation of period p^α operating in M and let L' be the totality of points which are invariant under T^q, $q = p^{\alpha-1}$. Since T^q is of period p, L' is non-empty by theorem I(1) and acyclic mod p by theorem II.

Moreover, L' is transformed into itself by T. The transformation induced in L' by T is the identity or else is periodic and of period p^β where $\beta < \alpha$. Hence T admits at least one fixed point in L'.

THEOREM III. *The hypothesis of finite dimensionality in theorems $I(\alpha)$ and II can be omitted if M is bicompact.*

The proof of theorem I(1) depends on the existence in a certain complex $\mathfrak{U}_m$ without fixed elements, of a sequence of cycles $C_0, C_1, \cdots$, terminating in a cycle of dimension greater than dim $\mathfrak{U}_m$. Suppose M is bicompact but not necessarily finite dimensional. Then, assuming that T has no fixed point, there exists a complete family $\{\mathfrak{U}\}$ of invariant coverings without fixed simplexes (although the dimensions of the $\mathfrak{U}$'s are not necessarily bounded). Moreover there can be described a construction whereby in any given $\mathfrak{U}$ there can be built a sequence $C_0, C_1, \cdots$, terminating with any desired dimension, in particular with a dimension greater than dim $\mathfrak{U}_m$. For, to say that M is acyclic now means simply that every cycle in M is ~ 0. Let $m = 1 + \dim \mathfrak{U}$ and let there be chosen refinements $\mathfrak{U} = \mathfrak{U}_m \supset \mathfrak{U}_{m-1} \supset \cdots \supset \mathfrak{U}_0$ such that the projection into $\mathfrak{U}_{i+1}$ of i-cycles in $\mathfrak{U}_i$ are ~ 0. Then we construct the desired cycles $C_0, \cdots, C_m$ precisely as in I(1) except that now $A_i = M$ $(i = 0, \cdots, m)$. Similar remarks apply to theorems II and $I(\alpha)$, $\alpha > 1$.

THEOREM IV. *Every periodic homeomorphic transformation of euclidean 3-space into itself admits at least one fixed point.*

Proof. Let E_3 be converted into a 3-sphere H_3 by the addition of a single point ∞. A periodic transformation operating in E_3 induces a periodic transformation which operates in H_3 and leaves ∞ fixed. It will be sufficient to show that the transformation of H_3 admits at least one fixed point different from ∞. This however follows from the fact that the fixed-point set of a periodic transformation operating in a 3-sphere, when it is not empty, consists of two points or else is homeomorphic to a circle or a 2-sphere [4].

6. We shall say that a transformation T of period q operating in a locally euclidean space is *regular* if those fixed-point sets of $T^2, \cdots, T^{q-1}$ which are not empty are locally euclidean. It can be shown for example that if T is locally analytic, it is regular.

THEOREM V. *A regular periodic homeomorphic transformation of euclidean 4-space into itself admits at least one fixed point.*

Proof. Let E_4 be converted into a 4-sphere by the addition of a single point ∞. Let q be the period of a regular transformation T operating in E_4.

We may suppose that q is not a power of 2 (theorem I(α)). Then $q = ps$, say, where p is an odd prime. Denoting also by T the transformation which is induced in H_4, let L' be the fixed point set of T^s in H_4. L' is not empty since it contains ∞. Since T^s is of prime period p, L' has the same modulo p Betti numbers as an r-sphere of 0, 1, 2 or 3 dimensions (a 0-sphere being a pair of points) ([3], page 160). Now r can be 3 only if T^s is of period 2 ([3], page 157) which is not the case. Thus L', being locally euclidean, is either a pair of points or a simple closed curve or a closed surface with the mod p Betti numbers of a 2-sphere. In this last case, L' would actually be homeomorphic to a 2-sphere. In any case L' is transformed into itself by T and the transformation induced in L' is either the identity or else is periodic and leaves ∞ fixed. The fixed-point set in L' is therefore a pair of points or a simple closed curve or a set of points, homeomorphic to a 2-sphere. Hence it contains at least one point other than ∞.

REFERENCES.

1. E. Čech, "Théorie générale de l'homologie dans un espace quelconque," *Fundamenta Mathematicae*, vol. 18 (1932), pp. 149-183.
2. P. A. Smith, "A theorem on fixed points for periodic transformations," *Annals of Mathematics*, vol. 35 (1934), pp. 572-578.
3. P. A. Smith, "Transformations of finite period," *Annals of Mathematics*, vol. 39 (1938), pp. 127-163.
4. P. A. Smith, "Transformations of finite period, II," *Annals of Mathematics*, vol. 40 (1939), pp. 690-711.

THE INSTITUTE FOR ADVANCED STUDY AND
BARNARD COLLEGE, COLUMBIA UNIVERSITY.

ON THE MATRIC EQUATION $TA = BT + C$.*

By MARK H. INGRAHAM and H. C. TRIMBLE.

The purpose of this paper is doubly two-fold. It first deals rationally with the matrix equation

$$TA = BT + C, \tag{1}$$

as an equation in the unknown matrix T. By finding the maximum rank of T when $C = 0$, a simple treatment of the similarity problem is given. Consideration of this equation for the case $A = B$, $C = 0$ leads naturally to an isomorphism noted by P. L. Trump,[1] N. Jacobson[2] and others. This isomorphism is used to give rationally a simple treatment of certain problems connected with the ring of matrices commutative with a given matrix. Moreover, the work is carried through for matrices whose elements belong to a division algebra of finite order over its centrum so that not only are certain classical results for fields simplified and extended but the theory is generalized to this important non-commutative case.

It has been difficult for the authors to decide on the proper mode of presentation of the theory. A treatment can be given that unifies both cases, but this is definitely less simple than that for the commutative case alone. Another possible method would be to present the commutative case as a unit first and then the non-commutative case. This results in much duplication. It was therefore decided to write the treatment of each section for the commutative case and then add a sub-section to point out the alterations that are necessary to extend the treatment to the case of elements belonging to a division algebra. In some sections this takes only a few sentences; in others, a somewhat more elaborate treatment is necessary. It is hoped that by omitting all sub-sections headed "non-commutative case" those interested chiefly in the classical case can secure a unified treatment of the theory for matrices with elements in a field, while those interested in the general case will by this treatment find a sharper analysis of the difference between the cases than by other methods of presentation.

It should be borne in mind that "commutative case" and "non-commutative case" refer to the case where the fundamental number system from

* Received February 13, 1939; Revised April 15, 1940.

[1] Trump [9] (p. 376).

[2] Jacobson [2] (p. 502).

which the elements of the matrices are chosen forms a field or a division algebra of finite order respectively.

Though both authors collaborated throughout, Sections I-V are chiefly the work of Mr. Ingraham while the results of Section VI are, in the main, due to Mr. Trimble and form a portion of his doctoral dissertation. The authors wish to acknowledge with thanks the suggestions and help of J. H. Bell, C. J. Everett, Jr., and G. W. Whaples.[3]

I. Introductory theory.

1. Commutative case. In a previous paper by M. H. Ingraham and M. C. Wolf [4] certain theory was developed which will be used throughout this paper, part of which will be given in outline here for the convenience of the reader. Somewhat more general results couched in terms of ideal theory, were secured by Jacobson.[5] Let A be a square $n \times n$ matrix with elements in a field K, and $\xi_1, \xi_2, \cdots, \xi_k$ form a set of vectors ($n \times 1$ matrices) with elements in K. The vectors $\xi_1, \xi_2, \cdots, \xi_k$ are said to be linearly independent relative to A if, whenever a set of polynomials g_i in K is such that $\Sigma g_i(A)\xi_i = 0$, it follows that $g_i(A)\xi_i = 0$. The linear extension, relative to A of a set of vectors $\xi_1, \xi_2, \cdots, \xi_k$, denoted by $L_A(\xi_1, \xi_2, \cdots, \xi_k)$, is the totality of vectors of the form $\Sigma g_i(A)\xi_i$, where the g_i are polynomials in K. Such a set is said to be linear relative to A and obviously is linear with respect to coefficients in K. If $\xi_1, \xi_2, \cdots, \xi_k$ are linearly independent relative to A, they are said to form a proper base relative to A for their linear extension relative to A.

If ξ is any vector, there exists one and only one polynomial g of minimal degree, with leading coefficient unity, such that $g(A)\xi = 0$. This polynomial is said to be minimally associated with ξ relative to A; it divides every polynomial f associated with ξ, i. e., every polynomial f such that $f(A)\xi = 0$.

It can be shown that, relative to A, a proper base $(\xi_1, \xi_2, \cdots, \xi_k)$ for the total vector space V may be found. Let g_i be minimally associated with ξ_i relative to A. A complete set of invariants for the system of such proper bases is given by

THEOREM 1.[6] *If g is irreducible, the number of g_i divisible by any power g^l of g is the same for all proper bases of V.*

---

[3] The work of Mr. Trimble and Mr. Everett has been made possible by grants from the Research Committee of the University of Wisconsin.

[4] Ingraham, Wolf [1].

[5] Jacobson [2].

[6] Ingraham, Wolf [1] (p. 20).

The ξ_i may be so chosen that the g_i are the invariant factors, the characteristic divisors, or the elementary divisors of A.

We shall denote the degree of any polynomial g by $< g >$. The g_i of the preceding paragraph satisfy the relation $\Sigma < g_i > = n$.

2. The non-commutative case. If the elements of A are not in a field but are in a division algebra D of finite order over its centrum C, the theory is somewhat more complicated and is based in part on the work of O. Ore [7] concerning non-commutative polynomials. If $g = \Sigma\lambda^i a_i$ is a polynomial with coefficients in D, A a matrix, and ξ a vector with elements in D, then $g(A) \odot \xi$ is defined to be $\Sigma A^i \xi a_i$. The products $g(A) \odot T$ and $g_1 \odot g_2$ are defined in a similar manner. The polynomial $g_1 \odot g_2$ in a commutative indeterminate is the $g_2 g_1$ as usually defined. An immediate extension of the notions [8] of relative linear extensions and relative linear independence to the $\odot$-process are available, as well as the idea of minimal association.

If $g = g_1 \odot g_2$, g_2 is said to be an interior factor of g. If g is any polynomial, then there exists a polynomial h of minimum degree with coefficients in the centrum and leading coefficient unity, such that h may be expressed in the form $f \odot g$, where f is a polynomial. This polynomial h divides all other polynomials with coefficients in the centrum having g as an interior factor. We say that g defines h. If h is defined by g_1 but by no proper factor of g_1, and if h is defined by g_2 but by no proper factor of g_2, then $< g_1 > = < g_2 >$, and this number, denoted by $<< h >>$, is said to be the reduced degree of h. If g is irreducible and defines h, then h is irreducible in the centrum, and conversely, if h is irreducible in the centrum, is defined by g, and $< g > = << h >>$, then g is irreducible.

The following three theorems [9] are restated for the convenience of the reader.

THEOREM 2. *If h is an irreducible polynomial, and if g_1 is a polynomial of minimum degree defining h, there exists a set of transforms g_i $(i = 1, 2, \cdots, t)$ of g_1 by the basal elements α_i, such that $g_t \odot g_{t-1} \odot \cdots \odot g_1$ is a polynomial of minimum degree defining h^t, and hence $<< h^t >> = t << h >>$.*

Let $g^{(t)} = g_t \odot g_{t-1} \odot \cdots \odot g_1$.

THEOREM 3. *If U is a relative linear set and if h is minimally associated with a vector ξ mod U over the centrum C relative to M and if g defines h,*

---

[7] Ore [5], [6].

[8] Ingraham, Wolf [1] (p. 22).

[9] Ingraham, Wolf [1], Theorems 25, 21, and 26.

but no interior factor of g of lower degree than the degree of g defines h, then there exists a vector η *in* $L_M(\xi)$ *such that:* (1) *g is minimally associated with* η *mod U, and* (2) *there exists a polynomial p such that* $g(M) \odot \eta = p(M) \odot h(M) \odot \xi$.

THEOREM 4. *If* $\xi_1, \xi_2, \cdots, \xi_k$ *is a set of vectors linearly independent relative to M, such that* $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ *is the whole space, and if* g_i *is the minimum polynomial associated with* ξ_i *relative to M, then the rank of* $h(M)$, *where h is any polynomial over the centrum, is equal to* $m - \sum_{i=1}^{k} < g_{1i} >$, *where m is the order of M and* $g_{1i} = (h, g_i)_{ex}$, *the greatest common exterior divisor of h and* g_i.

From these it follows that, relative to any square matrix M, a proper base $\xi_1, \cdots, \xi_k$ of the total vector space, V, may be found such that ξ_i is minimally associated with $g_i^{(t_i)}$, where g_i is irreducible.

It may be shown [10] that this may be done in such a way that $g_i^{(t)}(M) \odot \xi_i$ may be written in the form $p_{it}(M) \odot h_i^t(M) \odot \xi_i$, where h_i is the polynomial defined by g_i. Of course this may be done constructively only in algebras D for which there exists a constructive process for factoring a polynomial into irreducible factors.

II. The equation $TA = BT + C$.

1. Commutative case. D. E. Rutherford [11] considered this equation getting a complete solution, but his method involves the necessity of using the characteristic values of A and B. This equation was also studied by R. Weitzenböck,[12] who showed the existence of what he calls a "reine" solution. The following gives a rational construction for the complete rational solution.

Consider the equation $TA = BT + C$, where A is a square $n \times n$ matrix, B a square $m \times m$ matrix, and T and C $m \times n$ matrices.

Let ξ_i $(i = 1, \cdots, k_1)$ form a proper base relative to A for the total n-space, V_1, and let ξ_i be minimally associated with g_i.

Let η_i $(i = 1, \cdots, k_2)$ form a proper base for the total m-space, V_2, relative to B, and let η_i be minimally associated with f_i.

The matrix T will satisfy equation (1) if and only if

$$TA\xi = (BT + C)\xi \tag{2}$$

for every ξ of a set of n linearly independent vectors of order n. We may

[10] Ingraham, Wolf [1] (p. 28).
[11] Rutherford [7].
[12] Weitzenböck [11].

choose as such a set $A^l\xi_i$ $(i = 1, \cdots, k_1,\ l = 0, 1, \cdots, < g_i > - 1)$ since any linear relation between these would negate the hypothesis that $\xi_1, \xi_2, \cdots \xi_{k_1}$, were linearly independent relative to A.

From (1) it follows that

$$TA^2 = B^2T + BC + CA,$$

and in general,

$$TA^r = B^rT + B^{r-1}C + B^{r-2}CA + \cdots + BCA^{r-2} + CA^{r-1} \qquad (r = 1, 2, 3 \cdots).$$

Call

$$\mathfrak{D}_C(\lambda^r : B, A) = B^{r-1}C + B^{r-2}CA + \cdots + BCA^{r-2} + CA^{r-1},$$

and in particular

$$\mathfrak{D}_C(\lambda^1 : B, A) = C \text{ and } \mathfrak{D}_C(\lambda^0 : B, A) = 0.$$

We may then write

(3) $$TA^r = B^rT + \mathfrak{D}_C(\lambda^r : B, A).$$

If $g = \Sigma\lambda^i a_i$, we define $\mathfrak{D}_C(g : B. A)$ to be $\Sigma\mathfrak{D}_C(\lambda^i : B, A)a_i$. When no ambiguity exists, we will denote this merely by $\mathfrak{D}(g)$. From (3) it follows that

(4) $$Tg(A) = g(B)T + \mathfrak{D}(g).$$

Let $T\xi_i = \zeta_i$. Since $g_i(A)\xi_i = 0$, it follows from (4) that

$$0 = Tg_i(A)\xi_i = g_i(B)\zeta_i + \mathfrak{D}(g_i)\xi_i.$$

Hence the ζ_i satisfy the equations

(5) $$g_i(B)\zeta_i = -\mathfrak{D}(g_i)\xi_i.$$

Moreover, if we have a set ζ_i satisfying (5), we may determine T by setting

$$TA^l\xi_i = B^l\zeta_i + \mathfrak{D}(\lambda^l)\xi_i \qquad (i = 1, \cdots, k_1,\ l = 0, \cdots, < g_i > - 1).$$

so that

(6) $$(TA)A^{l-1}\xi_i = (BT + C)A^{l-1}\xi_i \qquad (i = 1, \cdots, k_1,\ l = 1, \cdots, < g_i > - 1).$$

Since equations (5) guarantee that

$$TA(A^{<g_i>-1})\xi_i = (BT + C)A^{<g_i>-1}\xi_i,$$

we see by using equation (6) that T satisfies equations (2) for the set of vectors $A^l\xi_i$ $(i = 1, \cdots, k_1,\ l = 0, \cdots, < g_i > - 1)$.

The problem is therefore reduced to the determination of ζ_i satisfying equations (5). Since the η_i form a proper base for V_2 relative to B, we may write

$$-\mathfrak{D}(g_i)\xi_i = \Sigma_j q_{ij}(B)\eta_j,$$

where the q_{ij} are polynomials over K. Let

$$\zeta_i = \Sigma_j x_{ij}(B)\eta_j,$$

where the x_{ij} are polynomials over K. It follows from equations (5) that the x_{ij} must satisfy the equations

$$g_i(B)x_{ij}(B)\eta_j = q_{ij}(B)\eta_j \qquad (i, j),$$

which is equivalent to saying

(7) $$g_i x_{ij} \equiv q_{ij} \bmod f_j,$$

a system of congruences whose solution is well known.

If T_0 satisfies equation (1) and if T is any other solution, then $S = T - T_0$ satisfies the equation

(8) $$SA = BS.$$

For this, equations (7) reduce to

(9) $$g_i x_{ij} \equiv 0 \bmod f_j.$$

Hence $x_{ij} = y_{ij} \dfrac{f_j}{(g_i, f_j)}$ where the y_{ij} may be reduced modulo the greatest common divisors (g_i, f_j) of g_i and f_j. From this it is clear that the number of linearly independent solutions of equation (8) is $\Sigma_{ij} <(g_i, f_j)>$, which, since the g_i and f_j may be taken to be the invariant factors of A and B respectively, gives the well known [13] result that the number of independent solutions of equation (8) is the sum of the degrees of the greatest common divisors of the invariant factors of A and B taken for all possible pairs. This also determines the number of independent solutions of equation (1).

2. Non-commutative case. This section up through equation (9) may be applied to the non-commutative case by merely replacing multiplication by the $\odot$-process.

We will indicate the nature of the solution of an equation of the type of (7). This is chiefly based on the work of Ore.

Consider the congruence

(10) $$g \odot x \equiv q \bmod f.$$

Ore [14] has shown that the solution of this congruence may be made to depend on finding solutions of the two congruences

---

[13] MacDuffee [4] (p. 90).

[14] Ore [5] (p. 253). Though this work is preceded by certain postulates that limit the application to "differential polynomials," nevertheless, as the author points out on p. 236, it may be applied to a non-commutative ring with an euclidean algorithm, and hence to this case.

(11) $$g \odot x \equiv 0 \bmod f$$
and
$$g \odot x_1 \equiv q_1 \bmod p$$
where $f = p \odot (q, f)$ and the polynomial q_1 is defined by the method of reduction. In other words, the solutions may be made to depend on equations of type (11), and on equations of type (10) where q and f are relatively prime. It is clear, moreover, that any two solutions of (10) differ by a solution of (11).

Let $[g_1, g_2]$ be the least common exterior multiple of g_1 and g_2 and let d be defined by the equation $[q, f] = d \odot q$. Ore[15] proves that a necessary and sufficient condition for the existence of a solution of equation (10), under the condition stated above, is that there exists a polynomial f_1 relatively prime to g such that
$$[g, f_1] = d \odot g.$$

If $[x, f] = f_x \odot x)$ it can be shown[16] that the solutions of (11) are those polynomials x which in the sense of Ore transform f into a divisor of g. Of course the trivial solution 0 always exists. It is clear that, if h has coefficients in the centrum and if x is a solution of (11), $h \odot x$ is also a solution. If there is a non-trivial solution, then a set of solutions $x_{01}, \cdots x_{0k}$ may be found which are linearly independent as to coefficients in the ring of polynomials over the centrum and such that every solution is of the form $\Sigma h_i \odot x_{0i}$ where the h_i are polynomials over the centrum. The number k need not exceed the order of the algebra D over its centrum C. Let x_{01} be a non-trivial solution of minimum degree. Let x_{02} be a solution of minimum degree, linearly independent of x_{01}, and in general let x_{0i} be a solution of minimum degree linearly independent of $x_{01}, \cdots x_{0i-1}$ as to coefficients in the ring of poly nomials over the centrum. The leading coefficient of x_{0i} must be linearly independent as to coefficients in the centrum of the leading coefficients of $x_{01}, \cdots x_{0i-1}$; for if not there would exist polynomials of the form $h_j = \lambda^m c_j$ such that c_j are in the centrum and $x_{0i} - \Sigma_1^{i-1} h_j x_{0j}$ is of lower degree than x_{0i}, in contradiction to minimal hypothesis concerning the degree of x_{0i}. Hence k is not greater than the order of D over C.

It is easily seen that if the polynomial h_g, defined by g, and the polynomial h_f, defined by f, are relatively prime, then (11) has no solution other than $x \equiv 0 \bmod f$.

To illustrate the above let D be the system of rational quaternions. Here, the congruence $(\lambda + 1) \odot x \equiv 0 \bmod \lambda + i$ has only the trivial solution $x \equiv 0$.

---

[15] Ore [5] (pp. 235-236).

[16] Ore [6] (p. 489, Theorem 11).

The congruence $(\lambda + i) \odot x \equiv 0 \bmod \lambda + i$ has the two linearly independent solutions 1 and i; the congruence $(\lambda + 1) \odot x \equiv 0 \bmod (\lambda + 1)$ has the four solutions $1, i, j, k$; and the congruence $(\lambda + 1) \odot (\lambda - i) \odot x \equiv 0 \bmod (\lambda + 1) \odot (\lambda + i)$ has as a fundamental set of solutions $j, k, \lambda + i, i\lambda - 1$.

III. Characteristic divisors of a matrix for relative linear subspaces, and for relative linear spaces modulo a relative linear subspace.

1. Commutative case. If V_1 is a linear subspace relative to a $n \times n$ matrix M of the total n-space V, then M defines a transformation on V_1 to V_1 and this transformation has associated with it various invariants such as invariant factors, characteristic divisors, elementary divisors, etc. Denote these as the invariant factors of M on V_1, characteristic divisors of M on V_1, etc. If the equalities arising in such definitions are replaced by congruences modulo a subspace V_2 of V_1 linear relative to M, then we may speak of the various invariants of M on V_1 modulo V_2.

The following discussion leads to an interesting treatment of the nature of the solution of the equation $TA = BT$.

THEOREM 5. *If U is a subset of V, where U and V are linear relative to M, then the characteristic divisors of M on U are divisors of the corresponding characteristic divisors of M on V.*

A corollary of this is

THEOREM 6. *If U is a relative linear subspace of the total vector space, then the characteristic divisors of M on U are divisors of the characteristic divisors of M.*

It is clearly sufficient to prove Theorem 5 for the case where the characteristic divisors of M are powers of one irreducible polynomial h. Let these be $h^{v_1}, h^{v_2}, \cdots, h^{v_k}$, and let the characteristic divisors of M on U be $h^{u_1}, h^{u_2}, \cdots, h^{u_{k_1}}$, where $v_i \geq v_{i+1}$, $u_i \geq u_{i+1}$.

Let V_l and U_l be the linear spaces in the total vector space V and U respectively which are orthogonal to h^l. U_{l-1} is the intersection of V_{l-1} and U_l. V_l contains both U_l and V_{l-1}.

Hence order $V_l \geq$ order $U(V_{l-1}, U_l)$

$$= \text{order } V_{l-1} + \text{order } U_l - \text{order } U_{l-1}$$

and

$$\text{order } V_l - \text{order } V_{l-1} \geq \text{order } U_l - \text{order } U_{l-1}.$$

The left-hand side of this equation is the number of $v_i \geq l$ and the right-hand side the number of $u_i \geq l$. Theorem 5 follows at once.

THEOREM 7. *If U is a subset of V, where U and V are linear relative to M, then the characteristic divisors of M on V modulo U are divisors of the characteristic divisors of M on V.*

It is sufficient to prove this for the case where the characteristic divisors are all powers of a single irreducible polynomial h. Let $\sigma_1, \sigma_2, \cdots \sigma_k$ form a proper base for V relative to M and be minimally associated with the characteristic divisors $h^{s_1}, h^{s_2}, \cdots, h^{s_k}$. We proceed to form relative to M a proper base $\tau_1, \tau_2, \cdots, \tau_{k_1}$, for V mod U. Let τ_1 be σ_{p_1} where σ_{p_1} is minimally associated mod U with the highest power h^{t_1} of h which is minimally associated mod U with any of the σ. Let τ_{21} be σ_{p_2} where σ_{p_2} is minimally associated mod $(U + L_M(\tau_1))$ with the highest power h^{t_2} of h which is minimally associated mod $(U + L_M(\tau_1))$ with any of the σ. Choose τ_2 to be a linear combination relative to M of τ_1 and τ_{21}, such that τ_{21} is in $L_M(\tau_1, \tau_2)$ and such that τ_1 and τ_2 are linearly independent relative to M mod U. In general, let τ_{i1} be σ_{p_i}, where σ_{p_i} is minimally associated mod $(U + L_M(\tau_1 \cdots \tau_{i-1}))$ with the highest power h^{t_i} of h which is minimally associated mod $(U + L_M(\tau_1 \cdots \tau_{i-1}))$ with any of the σ, and form τ_i a linear combination relative to M of τ_{i1} and $\tau_1 \cdots \tau_{i-1}$ such that τ_{i1} is in $L_M(\tau_1 \cdots \tau_i)$ and $\tau_1 \cdots \tau_i$ are linearly independent relative to M mod U. The existence of such τ_i was shown by Ingraham and Wolf.[17] This may be continued until a base relative to M for V mod U is obtained. From this construction it follows that 1) $\tau_1, \tau_2, \cdots, \tau_{k_1}$ form a proper base relative to M for V mod U; 2) $h^{t_1}, h^{t_2}, \cdots, h^{t_{k_1}}$ are the characteristic divisors of M mod U; 3) $t_i \leq s_{p_i}$; 4) $t_i \geq t_{i+1}$. Hence the number of t_i greater than any integer l is equal to or less than the number of s_i greater than l and hence $s_i \geq t_i$ and Theorem 7 follows.

We may of course add that if U_1, U_2 are subsets of V, where U_1, U_2 and V are linear relative to M, and U_1 contains U_2, then the characteristic divisors of M on V mod U_2 are divisors of the characteristic divisors of M on V mod U_2.

THEOREM 8. *If the polynomials $h_{11}, h_{12}, \cdots, h_{1k}$ are divisors of the characteristic divisors $h_1, h_2, \cdots, h_k$ of M on V, where V is linear relative to M, there exist subsets V_1 and U_1 of V such that V_1 and U_1 are linear relative to M, and such that the characteristic divisors of M on V_1 are equal to the characteristic divisors of M on V mod U_1, and are $h_{11}, h_{12}, \cdots, h_{1k}$.*

Let $(\sigma_1, \sigma_2, \cdots, \sigma_k)$ be a proper base for V relative to M and let σ_i

[17] Ingraham, Wolf [1] Part I.

be minimally associated with h_i relative to M. Let $h_i = h_{2i}h_{1i}$. Then $L_M[h_{21}(M)\sigma_1, h_{22}(M)\sigma_2, \cdots, h_{2k}(M)\sigma_k]$ is effective as V_1, and $L_M[h_{11}(M)\sigma_1, h_{12}(M)\sigma_2, \cdots, h_{1k}(M)\sigma_k]$ is effective as U_1.

Hence at once in light of Theorems 5 and 7 there is an automorphism of the subsets linear relative to M of a space V linear relative to M such that if $V_1 \sim U_1$, then the characteristic divisors of M on V_1 equal the characteristic divisors of M on V mod U_1. This automorphism is not necessarily $1 \leftrightarrow 1$.

2. Non-commutative case. In the proof of Theorem 5 for the non-commutative case the order of V_l can be shown to be the number of $v_i \geqq l$ times the reduced degree of h.

The extension of the proof of Theorem 7 can be made to depend on the following lemma:

LEMMA 1. *If a polynomial g minimally defines h, any factor of g minimally defines the polynomial which it defines.*

For any polynomial f defining a polynomial h, the reduced degree of f, $\langle\langle f \rangle\rangle$, is defined to be the reduced degree of h, $\langle\langle h \rangle\rangle$. It follows that $\langle\langle f \rangle\rangle \leq \langle f \rangle$.

Let $g = g_1 \odot g_2$ and let g_1 and g_2 define h_1 and h_2 respectively. The polynomial g is associated with the product $h_1 \odot h_2$. Hence by Theorem 2 and the readily proved fact that if g_1 minimally defines h_1, and g_2 minimally defines h_2 where $(h_1, h_2) = 1$, then $g_1 \odot g_2$ minimally defines $h_1 \odot h_2$, we have

$$(12) \qquad \begin{aligned} \langle\langle g \rangle\rangle &\leqq \langle\langle h_1 \odot h_2 \rangle\rangle = \langle\langle h_1 \rangle\rangle + \langle\langle h_2 \rangle\rangle \\ &\leqq \langle\langle g_1 \rangle\rangle + \langle\langle g_2 \rangle\rangle. \end{aligned}$$

Since $\langle\langle g_i \rangle\rangle \leq \langle g_i \rangle$ and by hypothesis

$$\langle\langle g \rangle\rangle = \langle g \rangle = \langle g_1 \rangle + \langle g_2 \rangle$$

it follows from (12) that $\langle g_i \rangle = \langle\langle g_i \rangle\rangle$. Lemma 1 is an immediate consequence. Using the notation of Theorem 7, if τ_j $(j < i)$ is minimally associated with h^{u_j} over C relative to M and is minimally associated with $g^{(u_j)}$ where $g^{(u_j)}$ is the polynomial defined in connection with Theorem 2, we let $U_i = U + L_M(\tau_1, \cdots, \tau_{i-1})$. Then if σ_{p_i} is minimally associated mod U_i with h^{u_i} over C relative to M it is minimally associated mod U_i with a polynomial of degree $\langle\langle h \rangle\rangle$ u_i relative to M. By Theorem 3 there exists in $L_M(\sigma_{p_i})$ a vector τ_i associated with $g^{(u_i)}$ and such that the order of $L_M(\tau_i)$ mod U_1 is the order of $L_M(\sigma_{p_i})$ mod U_1, $L_M(\tau_i) = L_M(\sigma_{p_i})$ mod U_1. The argument of the proof of Theorem 7 may be carried through with no other difficulty.

IV. The rank of T where $TA = BT$.

1. Commutative case. In this section another approach to the solution of the equation

$$TA = BT \tag{13}$$

is given. It does not lead as immediately to the construction of T as does that of Sec. II but perhaps gives more insight into the nature of T. Moreover, explicit results on the possible ranks of T are given.

As in Sec. II let V_1 be the total n-space and V_2 the total m-space. Let V_{11} be the space of all vectors ξ in V_1 such that $T\xi = 0$. V_{11} is linear relative to A.

Let $TV_1 = V_{21}$. Since $Tg(A)\xi = g(B)T\xi$, V_{21} is linear relative to B. Moreover $g(B)T\xi = 0$ if and only if $g(A)\xi \equiv 0 \bmod V_{11}$. Hence the characteristic divisors of A mod V_{11} are equal to the characteristic divisors of B on V_{21}. Moreover, if subsets V_{11} and V_{21} of V_1 and V_2 respectively, exist such that V_{11} is linear relative to A and V_{21} is linear relative to B, and the characteristic divisors of A mod V_{11} are equal to the characteristic divisors of B on V_{21}, then there exists a T such that $TA = BT$ where $TV_1 = V_{21}$ and $TV_{11} = 0$, for there exist vectors $\xi_1, \xi_2, \cdots, \xi_k$, linearly independent relative to A mod V_{11}, and associated with the characteristic divisors $h_1, h_2, \cdots, h_k$ of A mod V_{11} and there exist vectors $\zeta_1, \zeta_2, \cdots, \zeta_k$ linearly independent relative to B which form a base for V_{21}, and which are also associated with the characteristic divisors of B on V_{21}. Since $L_A(\xi_1, \xi_2, \cdots, \xi_k) + V_{11} = V_1$, we see that T is completely defined by $T\xi_i = \zeta_i$, $TV_{11} = 0$ and satisfies the stipulated conditions. The rank of T is clearly the order of V_{21}.

By the discussion following Theorem 8, for every subset V_{11} of V_1 linear relative to A, there exists a subset V_{12}, linear relative to A, such that the characteristic divisors of A on V_{12} equal the characteristic divisors of A mod V_{11}. Hence we conclude

THEOREM 9. *The possible ranks of matrices T satisfying $TA = BT$ are the orders of subspaces V_{12}, and V_{21}, linear relative to A and B respectively, such that the characteristic divisors of A on V_{12} equal the characteristic divisors of B on V_{21}.*

Let the highest power of an irreducible polynomial h appearing among the characteristic divisors of A correspond to the highest power of h among the characteristic divisors of B, the second highest, to the second highest, etc.

Then by Theorem 5, Theorem 9 yields

THEOREM 10. *The possible rank of a matrix T satisfying $TA = BT$ is the sum of the degrees of a possible set of common divisors of corresponding characteristic divisors (or invariant factors) of A and B.*

and hence we readily get

THEOREM 11. *The maximum rank for all matrices T satisfying $TA = BT$ is the sum of the degrees of the greatest common divisors of corresponding characteristic divisors (or invariant factors) of A and B.*

The fundamental theorem on the equivalence of the identity of invariant factors to similarity of matrices follows at once, since T is non-singular only if the rank of T equals the order of A and B.

Theorem 9 may be stated in the form

THEOREM 12. *The possible ranks of matrices T satisfying $TA = BT$ are equal to the orders of subspaces of V_1 and V_2 linear relative to A and B respectively on which A and B are similar.*

2. The Non-commutative case. This case presents no additional difficulties not cared for by Sec. II.

V. A system isomorphic to the ring of matrices commutative with a given matrix.

1. Commutative case. The following isomorphism in slightly different form was given for a special case by Trump,[18] where the g_i were the elementary divisors of A. Later a more general form than that required here was given by Jacobson.[19] As the derivation at this point is easy and gives explicitly the form of the matrices involved, it will be given.

If

$$T_rA = AT_r \qquad (r = 1, 2)$$

$$T_2T_1A = AT_2T_1 \text{ and } (T_1 + T_2)A = A(T_1 + T_2).$$

Let
$$T_r\xi_1 = \Sigma x_{ij}^{(r)}(A)\xi_j \qquad (r = 1, 2)$$
then
$$(T_1 + T_2)\xi_i = \Sigma(x_{ij}^{(1)}(A) + x_{ij}^{(2)}(A))\xi_j$$
and
$$T_2T_1\xi_i = \Sigma x_{ij}^{(1)}(A)x_{jk}^{(2)}(A)\xi_k.$$

Moreover, from equation (9) with f_j replaced by g_j we see that $x_{ij}^{(r)}$ must be divisible by $v_{ij} = \dfrac{g_j}{(g_i, g_j)}$, so that we may write $x_{ij}^{(r)} = y_{ij}^{(r)}v_{ij}$, and the

[18] Trump [9] (p. 376).

[19] Jacobson [2] (p. 502).

algebra of matrices commutative with A is isomorphic to the algebra of transposes of the matrices (x_{ij}), that is, the algebra of matrices of the form

$$X = (x_{ji} = y_{ji} v_{ji})$$

where the elements of the j-th row are reduced modulo g_j.

This matrix is particularly convenient in two cases: 1) when the g_i are the invariant factors of A and 2) when the g_i are the characteristic divisors of A. To illustrate: In the case of three invariant factors g_1, g_2, g_3, X is of the form

$$\begin{bmatrix} y_{11} & y_{12}\dfrac{g_1}{g_2} & y_{13}\dfrac{g_1}{g_3} \\ y_{21} & y_{22} & y_{23}\dfrac{g_2}{g_3} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$$

where the rows are reduced modulo g_1, g_2 and g_3 respectively. In the second case X is the direct sum of matrices of the form given below for the case of three characteristic divisors g^{r_1}, g^{r_2}, g^{r_3}, g being irreducible, $r_1 \geqq r_2 \geqq r_3$,

$$\begin{pmatrix} y_{11} & y_{12} g^{(r_1 - r_2)} & y_{13} g^{(r_1 - r_3)} \\ y_{21} & y_{22} & y_{23} g^{(r_2 - r_3)} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

the rows being reduced modulo g^{r_1}, g^{r_2} and g^{r_3} respectively. Note that A always corresponds to λI.

2. Non-commutative case. It can be shown [20] that the η's may be picked in such a way that the x_{ij} of equation (9) may be taken to contain the polynomial defined by f_j divided by the greatest common divisor of that polynomial and the polynomial defined by g_i. Let this be v_{ij}. Corresponding to T there is a matrix X of polynomials $x_{ij} = y_{ij} \odot v_{ij}$ where the y_{ij} may be suitably reduced. This form for x_{ij} is not sufficient to prove that (9) is satisfied. The nature of the extra conditions can be seen from the only case we will discuss, namely, where the g_i and g_j are $g^{(r_i)}$ and $g^{(r_j)}$, the g being irreducible. By a familiar argument all other cases reduce to this though the isomorphism thus produced may not always be the most convenient. We will need the following lemma:

LEMMA 2. *If g is irreducible and defines h and if $h^{t-u} \odot f$ is divisible by $g^{(t)}$, then f is divisible by $g^{(u)}$.*

---

[20] The proof involves some complications which hardly seem worth including in detail. Methods involve those given in Ingraham, Wolf [1] (p. 28).

Let g_2 be the greatest common divisor of f and $g^{(u)}$ and be of lower degree than $g^{(u)}$. Then there exist polynomials p and q such that

$$p \odot f + q \odot g^{(u)} = g_2$$

and hence

$$h^{t-u} \odot (p \odot f + q \odot g^{(u)}) = p \odot h^{t-u} \odot f + q \odot h^{t-u} \odot g^{(u)} = h^{t-u} \odot g_2.$$

Each term of the left-hand side of this equation is by hypothesis divisible by $g^{(t)}$ but this is not true of the right-hand term, $h^{t-u} \odot g_2$, since g_2 defines a lower power of h than h^u.

Moreover, if $g^{(r)} \odot f$ is divisible by $g^{(k)}$, then $g^{(r-t)} \odot f$ is divisible by $g^{(k-t)}$, for $h^t \odot g^{(r-t)} \odot f$ is divisible by $g^{(k)}$ and our statement follows from Lemma 2.

In the case now under consideration $v_{ij} = h^{r_j - m_{ij}}$ where $m_{ij} =$ smaller of r_i, r_j. If then

$$g^{(r_i)} \odot x_{ij} = g^{(r_i)} \odot y_{ij} \odot h^{(r_j - m_{ij})}$$

is to be divisible by $g^{(r_j)}$, $g^{(r_i)} \odot y_{ij}$ must be divisible by $g^{(m_{ij})}$. This is also sufficient.

Moreover, if the y's satisfy the above relation it follows that $g^{(m_{ij})} \odot y_{jk} = g^{(r_j - (r_j - m_{ij}))} \odot y_{jk}$ is divisible by $g^{(m_{jk} - r_j + m_{ij})}$ if $m_{jk} - r_j + m_{ij} > 0$. Hence $g^{(r_i)} \odot y_{ij} \odot y_{jk}$ is divisible by $g^{(m_{jk} + m_{ij} - r_j)}$ if $m_{jk} - r_j + m_{ij} > 0$.

Hence

$$g^{(r_i)} \odot x_{ij} \odot x_{jk} = g^{(r_i)} \odot y_{ij} \odot y_{jk} \odot h^{(r_j + r_k - m_{ij} - m_{jk})}$$

is divisible by $g^{(r_k)}$. It readily follows that the matrices T commutative with A are isomorphic to the transforms of matrices with elements

$$x_{ij} = y_{ij} \odot h^{(r_j - m_{ij})}$$

where the y_{ij} may be reduced modulo $g^{(m_{ij})}$ and where the y_{ij} are such that $g^{(r_i)} \odot y_{ij}$ is divisible by $g^{(m_{ij})}$.

VI. Some applications of the isomorphisms to a study of the ring of matrices commutative with *A*.

It is the wish of the authors to demonstrate the power of the foregoing approach as a working method. The characteristic divisor and invariant factor isomorphisms come naturally from a consideration of bases of relative linear sets and the condition on a matrix B that $BA = AB$.

Using these isomorphisms the standard propositions on the nature of any matrix commutative with A [21] are proved simply, and new results are obtained.

[21] Wedderburn [10].

Not only the form but also the properties of matrices commutative with A are treated below. Let $R(A)$ be the ring of matrices commutative with A.

1. The determination of the ring $R(A)$. The problem of finding the most general matrix commutative with A is ordinarily reduced, not in general rationally, to the case where A has just one characteristic value which is assumed without loss of generality to be zero. Then A is taken in Jordan canonical form with blocks

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

of l_i $(i = 1, 2, \cdots, k)$ rows and columns along the main diagonal where the elementary divisors of A are $\lambda^{l_1}, \lambda^{l_2}, \cdots, \lambda^{l_k}$ and $l_1 \geq l_2 \geq \cdots \geq l_k$.

As an example of the simplest application of the characteristic divisor isomorphism consider

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where (δ_1, δ_4) form a proper base relative to A for the space, and δ_1, δ_4 are minimally associated with λ^3, λ^2 respectively.

If B is any matrix such that $BA = AB$, then B corresponds to

$$\begin{pmatrix} a_{11} + b_{11}\lambda + c_{11}\lambda^2, & \lambda(a_{12} + b_{12}\lambda) \\ a_{21} + b_{21}\lambda, & a_{22} + b_{22}\lambda \end{pmatrix}.$$

Then

$$B\delta_1 = (a_{11}I + b_{11}A + c_{11}A^2)\delta_1 + (a_{21}I + b_{21}A)\delta_4$$
$$B\delta_4 = (a_{12}A + b_{12}A^2)\delta_1 + (a_{22}I + b_{22}A)\delta_4.$$

Now

$$A\delta_1 = \delta_2, \; A^2\delta_1 = \delta_3, \quad A\delta_4 = \delta_5.$$

It is readily checked that

$$B(\delta_1, A\delta_1, A^2\delta_1, \delta_4, A\delta_4) = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ b_{11} & a_{11} & 0 & a_{12} & 0 \\ c_{11} & b_{11} & a_{11} & b_{12} & a_{12} \\ a_{21} & 0 & 0 & a_{22} & 0 \\ b_{21} & a_{21} & 0 & b_{22} & a_{22} \end{pmatrix}$$

which also equals B in this case.

The procedure is general for any matrix A in Jordan canonical form, and with just one characteristic divisor which is taken to be zero.

Given a matrix A, it is not possible in general to find its characteristic values rationally in the field K. Wedderburn [22] gives a method due to Frobenius for finding B rationally, but criticizes the method because it fails to yield explicitly the form of B. An explicit solution of the problem, not wholly rational, was given by Rutherford.[23] From the present point of view the rational solution for B is essentially that given in the following example. Let A have the characteristic divisors

$$(\lambda^2+1)^2, \quad \lambda^2+1.$$

Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where A is not taken in any standard canonical form since we wish to indicate a more general procedure. To construct a proper base relative to A, consider δ_1 and δ_2.

$$A\delta_1 = \delta_6,\ A^2\delta_1 = -\delta_1 \text{ and hence } (A^2+I)\delta_1 = 0.$$
$$A\delta_2 = \delta_3 - \delta_5,\ A^2\delta_2 = \delta_4 - \delta_2,$$
$$A^3\delta_2 = -2\delta_3 + \delta_5,$$
$$A^4\delta_2 = -2\delta_4 + \delta_2, \text{ and hence } (A^4 + 2A^2 + I)\delta_2 = 0.$$

Since the linear extension relative to A of δ_1 consists of all vectors of the form

$$a_1\delta_1 + a_2\delta_6,$$

and the linear extension relative to A of δ_2 of all vectors of the form

$$b_1\delta_2 + b_2\delta_3 + b_3\delta_4 + b_4\delta_5$$

it follows that δ_1, δ_2 are linearly independent relative to A. Hence (δ_2, δ_1) form a proper base for the space relative to A, and δ_2 and δ_1 are minimally associated with $(\lambda^2+1)^2$ and λ^2+1 respectively which are therefore the characteristic divisors of A as stated above.

If B is any matrix such that $BA = AB$, then B corresponds to

[22] Wedderburn [10] (p. 106).
[23] Rutherford [7].

$$\begin{pmatrix} a_{11} + b_{11}\lambda + c_{11}\lambda^2 + d_{11}\lambda^3 & (\lambda^2+1)(a_{12}+b_{12}\lambda) \\ a_{21} + b_{21}\lambda & a_{22} + b_{22}\lambda \end{pmatrix}.$$

Then

$$B\delta_2 = (a_{11}I + b_{11}A + c_{11}A^2 + d_{11}A^3)\delta_2 + (a_{21}I + b_{21}A)\delta_1$$
$$B\delta_1 = (A^2 + I)(a_{12}I + b_{12}A)\delta_2 + (a_{22}I + b_{22}A)\delta_1.$$

It is readily checked that

$$BP = \begin{pmatrix} a_{21} & -b_{21} & -a_{21} & b_{21} & a_{22} & -b_{22} \\ a_{11} - c_{11} & -b_{11} + d_{11} & -a_{11} + c_{11} & b_{11} - d_{11} & 0 & 0 \\ b_{11} - 2d_{11} & a_{11} - 2c_{11} & -2b_{11} + 3d_{11} & -2a_{11} + 3c_{11} & -b_{12} & -a_{12} \\ c_{11} & b_{11} - 2d_{11} & a_{11} - 2c_{11} & -2b_{11} + 3d_{11} & a_{12} & -b_{12} \\ -b_{11} + d_{11} & -a_{11} + c_{11} & b_{11} - d_{11} & a_{11} - c_{11} & 0 & 0 \\ b_{21} & a_{21} & -b_{21} & -a_{21} & b_{22} & a_{22} \end{pmatrix}$$

where

$$P = (\delta_2, A\delta_2, A^2\delta_2, A^3\delta_2, \delta_1, A\delta_1) \text{ i.e.}$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence B is

$$\begin{pmatrix} a_{22} & a_{21} & 0 & 0 & b_{21} & -b_{22} \\ 0 & a_{11} - c_{11} & 0 & 0 & b_{11} - d_{11} & 0 \\ -b_{12} & b_{11} - 2d_{11} & a_{11} - c_{11} & -b_{11} + d_{11} & c_{11} & -a_{12} \\ a_{12} & c_{11} & b_{11} - d_{11} & a_{11} - c_{11} & d_{11} & -b_{12} \\ 0 & -b_{11} + d_{11} & 0 & 0 & a_{11} - c_{11} & 0 \\ b_{22} & b_{21} & 0 & 0 & -a_{21} & a_{22} \end{pmatrix}.$$

It would be quite possible [24] to follow the classical approach to a rational solution of the problem of determining the form of any matrix commutative with A, to reduce the problem by assuming the characteristic divisors of A are all powers of the same irreducible polynomial, and to begin with A in a convenient canonical form. Then there could be derived explicitly the form of any matrix commutative with this canonical form of A, and hence of any matrix commutative with A itself. But the problem of finding the characteristic divisors of A is essentially the same as that of setting up the isomorphism

[24] Williamson [12].

relative to A. It should be noted further that there is no need to determine the characteristic divisors of A, a process which requires factorization of polynomials in the field K into their irreducible factors. The invariant factors of A may be determined without such factorization of polynomials by the application of the greatest common divisor process, and then the invariant factor isomorphism may be used to find the most general matrix commutative with A.

2. Some properties of the ring $R(A)$. The proof of the following theorem is of interest, and may be compared with classical proofs such as that given by Wedderburn.[25]

THEOREM 13. *Any matrix which is commutative not only with A but also with every matrix commutative with A is a scalar polynomial in A.*

Let the matrices B in $R(A)$ correspond to (b_{ij}). We seek a matrix C in $R(A)$ such that for every B in $R(A)$, $BC = CB$.

If (b_{ij}) is the Kronecker delta matrix δ_{ij}, then it follows that

$$c_{ij} \equiv 0 \pmod{h^{l_i}} \qquad (i \neq j).$$

If (b_{ij}) is taken to be $\delta_{ik} h^{d_{ik}}$, where $d_{ik} = l_i - l_k$, $i < k$, it follows that

$$c_{kk} \equiv c_{ii} \pmod{h^{l_k}}$$

Hence $(c_{ij}) \equiv c_{11} I$ where by two matrices being congruent we shall mean that the corresponding elements in their i-th rows differ by multiples of h^{l_i} and corresponds to $c_{11}(A)$, where $c_{11}(\lambda)$ is a scalar polynomial.

The theorem of Sylvester: *If A is non-derogatory, then every matrix commutative with A is a polynomial in A,* is an obvious consequence of the invariant factor isomorphism. For A, being non-derogatory, has a single invariant factor, and if $BA = AB$ then B corresponds to $p(\lambda)$ where p is a scalar polynomial. Hence $B = p(A)$.

This theorem has an interesting generalization based upon the concept *of minimal congruences of a matrix in the isomorphic system.*

If $(\xi_1, \xi_2, \cdots, \xi_k)$ form a proper basis for the space relative to A, and if the ξ_i are minimally associated with h_i $(i = 1, 2, \cdots, k)$ respectively where the h_i are the invariant factors of A, then if $BA = AB$, suppose $G = (b_{ij})$ corresponds to B under the invariant factor isomorphism. Using small letters to denote polynomials in λ

$$h_1 I \equiv 0 \tag{14}$$

[25] Wedderburn [10] (pp. 105-106).

is a minimal relationship of the type to be defined. Since $h_1 G \equiv 0$, there is a relation $g_1 G + a'_{10} I \equiv 0$ such that g_1 is of minimal degree with leading coefficient unity. If $d = (g_1, h_1)$, and $mg_1 + nh_1 = d$, then $dG + ma'_{10} I \equiv 0$; hence $d = g_1$ divides h_1. Let $h_1 = g_1 q_1$. Then $h_1 G + q_1 a'_{10} I \equiv 0$, or $q_1 a'_{10} I \equiv 0$. Hence g_1 divides a'_{10}, and we may write $a'_{10} \equiv a_{10} g_1$ where a_{10} is reduced modulo q_1. The *minimal linear congruence* in G is defined to be

(15) $$g_1(G + a_{10} I) \equiv 0.$$

If $rG + sI \equiv 0$ it is readily checked that g_1 divides r, and g_1 divides s. Hence the reduced congruence (15) is unique.

Since $g_1 G^2 + g_1 a_{10} G \equiv 0$, there is a relation $g_2 G^2 + a'_{21} G + a'_{20} I \equiv 0$ such that g_2 is of minimal degree with leading coefficient unity. As before g_2 divides g_1. If $g_1 = g_2 q_2$, then $g_1 G^2 + q_2 a'_{21} G + q_2 a'_{20} I \equiv 0$ and by subtraction $(q_2 a'_{21} - g_1 a_{10}) G + q_2 a'_{20} I \equiv 0$. Hence g_1 divides $q_2 a'_{21} - g_1 a_{10}$, and g_1 divides $q_2 a'_{20}$. Then g_2 divides a'_{21} and g_2 divides a'_{20}. As before write $a'_{21} = g_2 a_{21}$, $a'_{20} = g_2 a_{20}$ where a_{21}, a_{20} are reduced modulo h_1/g_2. The *minimal quadratic congruence* in G is defined to be

(16) $$g_2(G^2 - a_{21} G - a_{20} I) \equiv 0.$$

The process may be continued to derive minimal congruences of higher degree in G. If G is considered as a matrix with elements in a commutative ring, apart from reductions and other special properties of the system, G satisfies its characteristic equation formed in the usual way as an identity. Hence G satisfies a congruence of degree at most its order and with leading coefficient unity. Thus

(17) $$G^r - a_{r,r-1} G^{r-1} - \cdots - a_{r1} G - a_{r0} I \equiv 0, \qquad r \leq k,$$

is called *the minimal congruence of* G if r is minimal. If coefficients are reduced by means of equations of lower degree, the equation is uniquely determined.

Under the isomorphism $a_{ij}(A)$ corresponds to a $a_{ij}(\lambda) I$, and hence in the original system the equations which correspond to the minimal congruences set up above are

(14′) $$h_1(A) = 0$$

(15′) $$g_1(A)[B - a_{10}(A)] = 0$$

(16′) $$g_2(A)[B^2 - a_{21}(A) B - a_{20}(A)] = 0$$

. . . .

(17′) $$B^r - a_{r,r-1}(A) B^{r-1} - \cdots - a_{r1}(A) B - a_{r0}(A) = 0, \qquad r \leq k.$$

Hence

THEOREM 14. *If A is an n-th order square matrix with elements in a commutative field K and with k invariant factors, and if B is any matrix in $R(A)$, then A and B satisfy equations of types* 14′ *to* 17′.

Corollary 1. *B satisfies an equation of degree at most k with leading coefficient unity and other coefficients polynomials in A over K.*

Since if A is non-derogatory, $k = 1$ and hence $B = p(A)$ where p is a polynomial over K, Sylvester's theorem is a consequence of Theorem 14.

UNIVERSITY OF WISCONSIN.

BIBLIOGRAPHY.

1. M. H. Ingraham and M. C. Wolf, "Relative linear sets and similarity of matrices whose elements belong to a division algebra," *Transactions of the American Mathematical Society*, vol. 42 (1937), pp. 16-31.

2. N. Jacobson, "Pseudo-linear transformations," *Annals of Mathematics*, vol. 38 (1937), pp. 484-507.

3. N. H. McCoy, "On quasi commutative matrices," *Transactions of the American Mathematical Society*, vol. 36 (1934), 2, pp. 327-340.

4. C. C. MacDuffee, "The theory of matrices," *Ergebnisse der Math.*, J. Springer, Berlin, 1933.

5. O. Ore, "Formale theorie der linearen differentialgleichungen II," *Journal für Math.*, vol. 168 (1932), pp. 233-252.

6. O. Ore, "Theory of non-commutative polynomials," *Annals of Mathematics*, vol. 34 (1933), pp. 480-508.

7. D. E. Rutherford, "On the solution of the matrix equation $AX + XB = C$," *Proc. of Sec. of Sciences, Koninklijke Akademie van Wetenschappen te Amsterdam*, vol. 35 (1932), 1, pp. 54-59.

8. D. E. Rutherford, "On the rational commutant of a square matrix," *Proc. Amsterdam*, vol. 35, pp. 870-875.

9. P. L. Trump, "On a reduction of a matrix by the group of matrices commutative with a given matrix," *Bulletin of the American Mathematical Society*, vol. 41 (1935), pp. 374-380.

10. J. H. M. Wedderburn, "Lectures on matrices," *American Mathematical Society Colloquium Publications*, vol. 17, pp. 102-114.

11. R. Weitzenböck, "Über die matrixgleichung $AX + XB = C$," *Proc. of Sec. of Sciences, Koninklijke Akademie van Wetenschappen te Amsterdam*, vol. 35 (1932), 1, pp. 60-61.

12. J. Williamson, "Idempotent and nilpotent elements of a matrix," *American Journal of Mathematics*, vol. 58 (1936), pp. 747-758.

ON THE DEGREE OF CONVERGENCE OF THE DERIVED SERIES OF BIRKHOFF.* [1]

By W. H. McEwen.

1. Introduction. Let $f(x)$ be a given function and let $S_N(x)$ $(N=1, 2,\cdots)$ represent the N-th order partial sums of its Birkhoff series, defined with respect to a given n-th order linear homogeneous differential system on an interval $0 \leqq x \leqq 1$ (or more generally $a \leqq x \leqq b$). For such sums Milne [2] has shown that if $f^{(m)}(x)$ is of limited variation on $(0, 1)$, where m is an arbitrary positive integer, and if $f, f', \cdots, f^{(m-1)}$ vanish at 0 and 1, then

$$f(x) - S_N(x) = O(1/N^m)$$

uniformly on $0 \leqq x \leqq 1$. The purpose of the present paper is to show (i) that Milne's result can be extended to the derivatives of $S_N(x)$, so that under the same hypotheses

$$f^{(k)}(x) - S_N^{(k)}(x) = O(1/N^{m-k})$$

uniformly on $0 \leqq x \leqq 1$, for $k=0, 1, \cdots, m-1$; (ii) that corresponding results for an interior interval $0 < \delta \leqq x \leqq 1-\delta$ may be obtained without assuming that $f, f', \cdots, f^{(m-1)}$ vanish at 0 and 1 if a suitable method of summation is used.

2. The sums $S_N^{(k)}(x)$. We begin with a brief description of the Birkhoff series. For more detailed information the reader is referred to one of the well known papers on the subject.[3]

Let the given n-th order differential system be

$$(1)\quad \begin{aligned} &L(u) + \lambda u \equiv u^{(n)} + P_2(x)u^{(n-2)} + \cdots + P_n(x)u + \lambda u = 0,\\ &W_j(u) = 0 \qquad (j=1, 2, \cdots, n), \end{aligned}$$

in which the functions $P_2, \cdots, P_n$ are continuous and have continuous derivatives of all orders on $(0, 1)$, and the boundary conditions (consisting of n linearly independent linear homogeneous forms in $u^{(j)}(0)$, $u^{(j)}(1)$, $j=0, 1, \cdots, n-1$) are normalized and regular.[4] Let $G(x, y:\lambda)$ be the Green's function

* Received January 6, 1940.

[1] Presented by title to the American Mathematical Society in September, 1938.

[2] W. E. Milne, *Transactions of the American Mathematical Society*, vol. 19 (1918), pp. 143-156.

[3] See G. D. Birkhoff, *Transactions of the American Mathematical Society*, vol. 9 (1908), pp. 373-395; J. Tamarkin, *Rendiconti del Circolo Matematico di Palermo*, vol. 34 (1912), pp. 345-382; M. H. Stone, *Transactions of the American Mathematical Society*, vol. 28 (1926), pp. 695-761. The paper by Stone is the one that the author has followed particularly.

[4] For definition of these terms see Birkhoff, *loc. cit.*, p. 382.

of the system. The poles of G are then the characteristic values of (1), and these form an infinite sequence of complex numbers $\{\lambda\}$ which may be arranged so that $|\lambda_1| \leqq |\lambda_2| \leqq \cdots$, $\lim_{\nu\to\infty} |\lambda_\nu| = \infty$. Let $\{R_i(x,y)\}$ be the corresponding sequence of residues of G. The Birkhoff series for $f(x)$ may then be written

$$\sum_{i=1}^{\infty} \int_0^1 f(y) R_i(x,y)\, dy.$$

If the poles are all simple [5] this has the form

$$\sum_{i=1}^{\infty} a_i u_i(x), \qquad a_i = \int_0^1 f(y) v_i(y)\, dy \Big/ \int_0^1 u_i(y) v_i(y)\, dy,$$

where $\{u_i(x)\}$ and $\{v_i(x)\}$ are the sequences of characteristic solutions of system (1) and its adjoint respectively.

Let $C_1, C_2, \cdots$ be a system of concentric circles in the complex λ-plane with centres at $\lambda = 0$ and radii $\Lambda_1, \Lambda_2, \cdots$ where $\Lambda_1 < \Lambda_2 < \cdots$, $\lim_{\nu\to\infty} \Lambda_\nu = \infty$. Let these circles be so drawn as to remain uniformly away from the poles of G [6] and such that between every two consecutive ones there is at least one pole of G and as few others as possible. Let N denote the number of poles (each counted according to its multiplicity) enclosed in any specified circle C_ν. Then the partial sums of the Birkhoff series may be written

$$(2) \qquad S_N(x) = 1/2\pi i \int_{C_\nu} \int_0^1 f(y) G(x, y; \lambda)\, dy\, d\lambda, \qquad i = \sqrt{-1}.$$

A more useful form of (2) is obtained by placing $\lambda = \rho^n$. The entire λ-plane is thus made to correspond to a sector Σ in the ρ-plane, composed of two adjacent sectors of the following set of $2n$ equal sectors:

$$S;\ l\pi/n \leqq \arg \rho \leqq (l+1)\pi/n \qquad (l = 0, 1, 2, \cdots, 2n-1).$$

The circles C_ν are then transformed into arcs of circles $|\rho| = R = (\Lambda_\nu)^{1/n}$ lying in the sector Σ. Let Γ denote these arcs. Then we can write

$$S_N(x) = 1/2\pi i \int_\Gamma \int_0^1 f(y) n\rho^{n-1} G(x, y; \rho^n)\, dy\, d\rho,$$

and the derivatives of S_N of arbitrary order k may be written

$$S^{(k)}{}_N(x) = 1/2\pi i \int_\Gamma \int_0^1 f(y) n\rho^{n-1} \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} dy\, d\rho.[7]$$

In connection with the circles $|\rho| = R$ it is to be noted that $R = O(N)$.

---

[5] The poles are in general simple when $|\lambda_i|$ is large, being always so if the system is of odd order, or is of even order and self-adjoint. Multiple poles if they occur are double.

[6] This is always possible.

[7] See Stone, *loc. cit.*, p. 741. The notation $\{A; B\}$ is used to indicate that A is to be taken when $x > y$ and B when $x < y$.

3. The expressions $\left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\}$. Asymptotic formulas for the derivatives of G of arbitrary order k are given in Stone's paper [8] (which paper will be referred to hereafter as (S)). In dealing with these formulas it is necessary to consider separately the two cases when the system (1) is of odd or of even order. For the sake of brevity we shall deal only with the case of odd order

$$n = 2\mu - 1,$$

and merely remark that the treatment of the case of even order is entirely analogous.

The formulas involve in an important way the n-th roots of -1. Let these be denoted by $\omega_1, \omega_2, \cdots, \omega_n$ and suppose for each sector S the subscripts are so distributed that for values of ρ in that sector

$$Re(\rho\omega_1) \leqq Re(\rho\omega_2) \leqq \cdots \leqq Re(\rho\omega_n), \qquad Re \equiv \text{“the real part of.”}$$

Consider one of the sectors S which make up Σ, and let γ denote the half of the arc Γ lying in it. Then when ρ is on γ, $Re(\rho\omega_i)$ is negative if $i < \mu$ and positive if $i > \mu$, whereas $Re(\rho\omega_i)$ changes sign if $i = \mu$, being negative on one half of γ and positive on the other, vanishing at the mid-point. Let γ_1 and γ_2 denote the two halves of γ on which $Re(\rho\omega_\mu)$ is negative and positive respectively.

From $(S$, p. 745) with the help of $(S$, Theorem III′, p. 706) we obtain the following formula which holds when ρ is on γ_1:

$$
\begin{aligned}
n\rho^{n-1}\left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} = \Bigg\{ & -\sum_{i=1}^{\mu} \omega_i e^{\rho\omega_i(x-y)} (\rho\omega_i)^k \left(1 + \sum_{s=1}^{m} \frac{M_{ks}(x,y)}{(\rho\omega_i)^s} + \frac{E(x,y,\rho)}{\rho^{m+1}}\right); \\
& \sum_{i=\mu+1}^{n} \omega_i e^{\rho\omega_i(x-y)} (\rho\omega_i)^k \left(1 + \sum_{s=1}^{m} \frac{M_{ks}(x,y)}{(\rho\omega_i)^s} + \frac{E(x,y,\rho)}{\rho^{m+1}}\right) \Bigg\} \\
& + \frac{\Delta_1^{(k)}}{[\theta_0] + e^{\rho\omega_\mu}[\theta_1]}, \qquad \left(\mu = \frac{n+1}{2}\right).
\end{aligned}
$$

As stated in footnote 7 the notation $\{A ; B\}$ is used to indicate that A is to be taken when $x > y$, and B when $x < y$. The integer m is arbitrary, and in section 5 will be identified with the m of the theorem there stated. The functions M_{ks} may be taken as continuous and differentiable to all orders on $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and moreover may be chosen independent of the particular sector S. E is used here, and elsewhere in this paper, to indicate any function which is uniformly bounded as $\rho \to \infty$. The particular nature of the expressions in the last term will be made clear later on in section 5.

For convenience in writing let

---

[8] Stone, *op. cit.*

$$\{g_{k0}\,;\,g_{k0}\} = \{-\sum_{i=1}^{\mu} \omega_i e^{\rho\omega_i(x-y)}(\rho\omega_i)^k;\ \sum_{i=\mu+1}^{n} \omega_i e^{\rho\omega_i(x-y)}(\rho\omega_i)^k\},$$

$$\{g_{ks}\,;\,g_{ks}\} = \{-\sum_{i=1}^{\mu} \omega_i e^{\rho\omega_i(x-y)}(\rho\omega_i)^k M_{ks};\ \sum_{i=\mu+1}^{n} \omega_i e^{\rho\omega_i(x-y)}(\rho\omega_i)^k M_{ks}\},$$

$(s = 1, 2, \cdots, m)$. Then, when ρ is on γ_1,

$$n\rho^{n-1}\left\{\frac{\partial^k G}{\partial x^k};\ \frac{\partial^k G}{\partial x^k}\right\} = \sum_{s=0}^{m}\{g_{ks}\,;\,g_{ks}\} + \frac{\Delta_1^{(k)}}{[\theta_0] + e^{\rho\omega_\mu}[\theta_1]} + O\left(\frac{1}{\rho^{m-k+1}}\right).$$

When ρ is on γ_2 the formula is similar except for a change in the ranges of the summations involved, which now must be $(1, \mu - 1)$ and (μ, n). The corresponding formulas for the other sector S of Σ are exactly the same (with the same M_{ks}'s), but with a new distribution of the subscripts of the ω's.

4. Preliminary lemma. The following result, which was used by Milne, will be required in the next section.

LEMMA. *If $F(y)$ is of limited variation on (a, b) and c is any constant $\neq 0$, then in the half-plane where $Re(c\rho) \leqq 0$,*

$$\int_\alpha^\beta e^{c\rho y} F(y)\,dy = O(1/\rho), \qquad (0 \leqq a \leqq \alpha \leqq \beta \leqq b \leqq 1).$$

It may be proved as follows:

$$\begin{aligned}\left|\int_\alpha^\beta e^{c\rho y}F(y)\,dy\right| &= \left|\frac{1}{c\rho}\int_\alpha^\beta F(y)\ d(e^{c\rho y})\right| \\ &\leqq \left|\frac{1}{c\rho}\left[e^{c\rho\beta}F(\beta) - e^{c\rho\alpha}F(\alpha) - \int_\alpha^\beta e^{c\rho y} d(F(y))\right]\right| \\ &\leqq \frac{1}{|c\rho|}\left[|F(\beta)| + |F(\alpha)| + \int_\alpha^\beta e^{Re(c\rho)} d(V(y))\right] \\ &\leqq \frac{1}{|c\rho|}\left[|F(\beta)| + |F(\alpha)| + V(\beta) - V(\alpha)\right]\end{aligned}$$

where $V(y)$ is the total variation of $F(y)$ in (a, y).

5. Degree of convergence of $S_N^{(k)}(x)$. In this section we prove the first main result of the paper.

THEOREM I. *Let m be an arbitrary positive integer, and suppose $f^{(m)}(x)$ is of limited variation on $(0, 1)$ and $f, f', \cdots, f^{(m-1)}$ vanish at 0 and 1. Then*

$$f^{(k)}(x) - S_N^{(k)}(x) = O\left(\frac{1}{N^{m-k}}\right)$$

uniformly on $0 \leqq x \leqq 1$, for $k = 0, 1, \cdots, m - 1$.

Consider first the integral

$$\int_0^1 f(y)\{g_{k0}, g_{k0}\}dy = -\sum_{i=1}^{\mu} \omega_i(\rho\omega_i)^k \int_0^x e^{\rho\omega_i(x-y)}f(y)\,dy + \sum_{i=\mu+1}^{n} \omega_i(\rho\omega_i)^k \int_x^1 e^{\rho\omega_i(x-y)}f(y)\,dy.$$

On integrating by parts m times, and using the hypothesis that $f, f', \cdots, f^{(m-1)}$ vanish at 0 and 1, we obtain

$$\int_0^1 f(y)\{g_{k0}, g_{k0}\}dy = -\sum_{i=1}^{\mu} \omega_i(\rho\omega_i)^{k-m} \int_0^x e^{\rho\omega_i(x-y)}f^{(m)}(y)\,dy$$
$$+ \sum_{i=\mu+1}^{n} \omega_i(\rho\omega_i)^{k-m} \int_x^1 e^{\rho\omega_i(x-y)}f^{(m)}(y)\,dy$$
$$+ f(x)\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-1} + f'(x)\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-2} + \cdots + f^{(m-1)}(x)\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-m}.$$

But, by hypothesis, $f^{(m)}$ is of limited variation on $(0,1)$. Hence, by an application of the lemma,

$$\int_0^1 f(y)\{g_{k0}, g_{k0}\}dy = O\left(\frac{1}{\rho^{m-k+1}}\right) + f(x)\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-1} + \cdots + f^{(m-1)}(x)\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-m}.$$

Next, consider the integral $\int_0^1 f(y)\{g_{k1}, g_{k1}\}dy$. Let functions $\psi_{ks}(x,y)$ be defined as follows:

$$\psi_{ks}(x,y) \equiv f(y)M_{ks}(x,y), \qquad (s = 1, 2, \cdots, m).$$

Then, under the hypothesis of the theorem, it is obvious that $\psi_{ks}, \psi'_{ks}, \cdots, \psi_{ks}{}^{(m-1)}$ vanish at $y = 0$ and $y = 1$, where the differentiations are in each case with respect to the second argument, the variable y. Also, $\psi_{ks}{}^{(m)}$, $\psi_{ks}{}^{(m-1)}$, etc. are of limited variation on $(0,1)$. Hence, on integrating by parts $m-1$ times and applying the lemma, we obtain

$$\int_0^1 f(y)\{g_{k1}, g_{k1}\}dy = O\left(\frac{1}{\rho^{m-k+1}}\right) + \psi_{k1}(x,x)\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-2} + \cdots + \psi_{k1}{}^{(m-2)}(x,x)\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-m}.$$

Proceeding in this manner, integrating by parts $m-2$ times in the case of g_{k2}, $m-3$ times in the case of g_{k3}, and so on, and then combining the various results, we find that

$$(3)\quad \int_0^1 f(y)\sum_{s=0}^{m} \{g_{ks}; g_{ks}\}dy = O\left(\frac{1}{\rho^{m-k+1}}\right) + f(x)\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-1}$$
$$+ (f'(x) + \psi_{k1}(x,x))\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-2}$$
$$+ \cdots$$
$$+ (f^{(m-1)}(x) + \psi_{k1}{}^{(m-2)}(x,x) + \cdots + \psi_{km}(x,x))\sum_{i=1}^{n} \omega_i(\rho\omega_i)^{k-m}.$$

Next, consider the integral

$$\int_0^1 \frac{\Delta_1^{(k)}}{[\theta_0]+e^{\rho\omega_\mu}[\theta_1]} f(y)\,dy = \frac{1}{[\theta_0]+e^{\rho\omega_\mu}[\theta_1]} \int_0^1 \Delta_1^{(k)} f(y)\,dy.$$

A description of the determinant $\Delta_1^{(k)}$ is given in (S, p. 745 and p. 717). For our purpose it will be sufficient to observe that the variable y occurs only in the elements of the last column of $\Delta_1^{(k)}$, and all the elements in the corresponding co-factors are uniformly bounded as $\rho \to \infty$, except for those in the first row of $\Delta_1^{(k)}$ which are all $O(\rho^k)$. Let $0, Y_1, \cdots, Y_n$ denote the elements of the last column, and let $X_0, X_1, X_2, \cdots, X_n$ be the corresponding co-factors. Then

$$\int_0^1 \Delta_1^{(k)} f(y)\,dy = \sum_{j=1}^{n} X_j \int_0^1 Y_j f(y)\,dy,$$

where (from S, p. 717 with the help of Theorem III′, p. 706)

$$(4)\qquad Y_j \equiv O\left(\frac{1}{\rho^{m+1}}\right) + \left(\sum_{i=1}^{\mu} e^{\rho\omega_i(1-y)}\beta_j\omega_i^{k_j+1} - \sum_{i=\mu+1}^{n} e^{-\rho\omega_i y}\alpha_j\omega_i^{k_j+1}\right)$$
$$+ \sum_{s=1}^{m}\left(\sum_{i=1}^{\mu} e^{\rho\omega_i(1-y)}\frac{N_{is}(y)}{\rho^s} - \sum_{i=\mu+1}^{n} e^{-\rho\omega_i y}\frac{N_{is}(y)}{\rho^s}\right).$$

Here the functions N_{is} have the same general properties as the functions M_{ks}, and the numbers α_j, β_j, k_j are constants in the boundary conditions of the differential system.

On multiplying (4) by $f(y)dy$ and integrating from 0 to 1, integrating by parts m times in the case of the first expression in parenthesis, $m-1$ times in the case of the term $s=1$, and so on, we find, after applying the lemma, that

$$\int_0^1 Y_j f(y)\,dy = O\left(\frac{1}{\rho^{m+1}}\right).$$

Hence, since $X_j = O(\rho^k)$, and since the expression $[\theta_0] + e^{\rho\omega_\mu}[\theta_1]$ is known to be uniformly bounded away from zero as $\rho \to \infty$,[9] we have

$$(5)\qquad \int_0^1 \frac{\Delta_1^{(k)}}{[\theta_0]+e^{\rho\omega_\mu}[\theta_1]} f(y)\,dy = O\left(\frac{1}{\rho^{m-k+1}}\right).$$

On combining (5) and (3), we obtain

$$(6)\qquad \int_0^1 n\rho^{n-1}\left\{\frac{\partial^k G}{\partial x^k};\frac{\partial^k G}{\partial x^k}\right\} f(y)\,dy = O\left(\frac{1}{\rho^{m-k+1}}\right) f(x)\sum_{i=1}^{n}\omega_i(\rho\omega_i)^{k-1}$$
$$+\cdots$$
$$+\ (f^{(k)}(x) + \psi_{k1}^{(k-1)}(x,x) + \cdots + \psi_{kk}(x,x))\sum_{i=1}^{n}\omega_i(\rho\omega_i)^{-1}$$
$$+\cdots$$
$$+\ (f^{(m-1)}(x) + \psi_{k1}^{(m-2)}(x,x) + \cdots + \psi_{km}(x,x)\sum_{i=1}^{n}\omega_i(\rho\omega_i)^{k-m}.$$

[9] See Birkhoff, *op. cit.*

Although this result was obtained for the case when ρ is on γ_1, it holds equally when ρ is on γ_2, and since the M_{ks}'s and therefore also the ψ_{ks}'s are independent of the sector S it holds generally when ρ is on Γ.

Finally, let us multiply (6) by $(1/(2\pi i))d\rho$ and integrate over the arc Γ on which $|\rho| = R$. In this connection we observe that if s is any integer, positive, negative or zero, the integral

$$\int_\Gamma \sum_{i=1}^{n} (\rho\omega_i)^s d(\rho\omega_i) = \sum_{i=1}^{n} \int_\Gamma (\rho\omega_i)^s d(\rho\omega_i)$$

is zero in every case except $s = -1$, when it has the value $2\pi i$. For, on setting $z = \rho\omega_i$ the integral may be expressed in the form

$$\int z^s dz,$$

where the path of integration is now the entire circumference of the circle $|z| = R$. Hence we can write

$$S_N^{(k)}(x) = \frac{1}{2\pi i}\int_\Gamma \int_0^1 n\rho^{n-1} \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} f(y)\,dy\,d\rho$$
$$= O\left(\frac{1}{R^{m-k}}\right) + f^{(k)}(x) + (\psi_{k1}{}^{(k-1)}(x,x) + \cdots + \psi_{kk}(x,x)).$$

But the functions $(\psi_{k1}{}^{(k-1)}(x,x) + \cdots + \psi_{kk}(x,x))$ are independent of ρ (and therefore of N), and hence must be identically zero if the sums $S_N^{(k)}(x)$ are to converge to $f^{(k)}(x)$ at all. Moreover, it will be observed that these functions are linear combinations of the functions $f(x), f'(x), \cdots, f^{(m-1)}(x)$, with coefficients depending only on the functions $M_{ks}(x,x), \cdots, M_{ks}{}^{(m-1)}(x,x)$. These coefficients, therefore, must be identically zero if there is to be convergence at all. But the fact of convergence has been established by the author,[10] under hypotheses on $f(x)$ which are substantially more general than those admitted in the theorem of this section. Hence we conclude that the coefficients in question are identically zero. This property of vanishing must be regarded as something inherent in the Green's function of the problem, and not in any way dependent on the functions $f(x)$. The author has verified this fact by direct calculation in the case $k = 1$. Thus, finally, since $R = O(N)$, we obtain

---

[10] W. H. McEwen, *Bulletin of the American Mathematical Society*, August, 1939, pp. 576-582; Theorem 2, p. 582. Here it is shown that if $f(x)$ satisfies the boundary conditions $W_j(f) = 0$, $W_j(L(f)) = 0, \cdots, W_j(L^{p-1}(f)) = 0$, $(j = 1, 2, \cdots, n)$ and if $L^p(f)$ is integrable, p being an arbitrary positive integer, then $\lim S_N^{(k)} = f^{(k)}$ uniformly on $(0,1)$, for $k = 0, 1, \cdots, pn - 1$. With p suitably chosen there will exist a class of arbitrary functions $f(x)$ satisfying the hypotheses of both this theorem and the one in the text. Hence for all such functions convergence is assured.

$$S_N^{(k)}(x) = f^{(k)}(x) + O\left(\frac{1}{N^{m-k}}\right)$$

uniformly on $0 \leqq x \leqq 1$, for $k = 0, 1, \cdots, m-1$, which proves the theorem.

6. A summation method. Stone [11] has indicated a method of summing the derived series of Birkhoff (essentially the method of Riesz typical means), in which the N-th order sums take the form

$$\sum_{i=1}^{N} \left(1 - \frac{\lambda_i^4}{\Lambda_\nu^4}\right)^{k+l} \int_0^1 f(y) \frac{\partial^k}{\partial x^k} R_i(x, y) dy, \qquad l \geqq 0, k = 0, 1, 2, \cdots.$$

We shall adapt this to our present needs (with slight modifications) to read as follows:

$$\sum_{i=1}^{N} \left(1 - \frac{\lambda_i^{4a}}{\Lambda_\nu^{4a}}\right)^m \int_0^1 f(y) \frac{\partial^k}{\partial x^k} R_i(x, y) dy \tag{8}$$

where, for reasons which will appear presently, a is assumed to be any positive integer such that $4an \geqq m$. Denoting by $\sigma_N(x)$ the sums (8) corresponding to the case $k = 0$, we are thus led to consider the sums $\sigma_N^{(k)}(x)$ for $k = 0, 1, \cdots, m-1$, which may be represented by the contour integral formula

$$\sigma_N^{(k)}(x) = \frac{1}{2\pi i} \int_\Gamma \left(1 - \frac{\rho^{4an}}{R^{4an}}\right)^m \int_0^1 n\rho^{n-1} \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} f(y) dy\, d\rho, \quad i = \sqrt{-1}.$$

7. Degree of convergence of $\sigma_N^{(k)}(x)$. In this section we discard from the hypothesis the requirement that $f, f', \cdots, f^{(m-1)}$ vanish at 0 and 1, and prove the following result.

THEOREM II. *If $f^{(m)}(x)$ is of limited variation on $(0, 1)$, then*

$$f^{(k)}(x) - \sigma_N^{(k)}(x) = O\left(\frac{1}{N^{m-k}}\right)$$

uniformly on $0 < \delta \leqq x \leqq 1 - \delta$, for $k = 0, 1, \cdots, m-1$.

Under the hypothesis of this theorem the asymptotic formula (6) will again represent

$$\int_0^1 n\rho^{n-1} \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} f(y) dy,$$

provided certain additional terms are now added to it. These terms arise from the various operations of integration by parts because of the failure of $f, f', \cdots, f^{(m-1)}$ to vanish at 0 and 1. Let $\Phi(x, \rho)$ denote these terms. Then, when ρ is on γ_1, we have explicitly

[11] Stone, *op. cit.*

$$\begin{aligned}\Phi(x,\rho) \equiv f(0) \sum_{i=1}^{\mu} \omega_i(\rho\omega_i)^{k-1}e^{\rho\omega_i x} - f(1)\sum_{i=\mu+1}^{n}\omega_i(\rho\omega_i)^{k-1}e^{\rho\omega_i(x-1)}\\ -(f'(0)+\psi_{k1}(x,0))\sum_{i=1}^{\mu}\omega_i(\rho\omega_i)^{k-2}e^{\rho\omega_i x}\\ -(f'(1)+\psi_{k1}(x,1))\sum_{i=\mu+1}^{n}\omega_i(\rho\omega_i)^{k-2}e^{\rho\omega_i(x-1)}\\ -\cdots\\ -(f^{(m-1)}(0)+\psi_{k1}{}^{(m-2)}(x,0)+\cdots+\psi_{km}(x,0))\sum_{i=1}^{\mu}\omega_i(\rho\omega_i)^{k-m}e^{\rho\omega_i x}\\ -(f^{(m-1)}(1)+\cdots+\psi_{km}(x,1))\sum_{i=\mu+1}^{n}\omega_i(\rho\omega_i)^{k-m}e^{\rho\omega_i(x-1)}.\end{aligned}$$

On multiplying $\Phi(x,\rho)$ by $(1/(2\pi i))(1-\rho^{4an}/R^{4an})^m d\rho$ and integrating over γ_1, we obtain the following set of integrals:

$$\int_{\gamma_1}\rho^{k-j}\left(1-\frac{\rho^{4an}}{R^{4an}}\right)^m e^{\rho\omega_i x}d\rho,\ (i=1,\cdots,\mu);$$
$$\int_{\gamma_1}\rho^{k-j}\left(1-\frac{\rho^{4an}}{R^{4an}}\right)^m e^{\rho\omega_i(x-1)}d\rho,\ (i=\mu+1,\cdots,n),$$

where $j=1,2,\cdots,m$. But with x restricted to $0<\delta\leqq x\leqq 1-\delta$, each of these integrals for which $i\neq\mu$ converges to zero to an infinite order in $1/R$ as $R\to\infty$ (since in each case the real part of the exponential factor is <0). On the other hand, when $i=\mu$, we can easily show, following the method of (S, p. 715, in the proof of Lemma V), that when x is thus restricted

$$\left|\int_{\gamma_1}\rho^{k-j}\left(1-\frac{\rho^{4an}}{R^{4an}}\right)^m e^{\rho\omega_\mu x}d\rho\right|\leqq\frac{K}{R^{m-k+j}}=O\left(\frac{1}{N^{m-k+j}}\right).$$

The proof is as follows. Let ϕ be an angle measured from the bisecting ray of S (positively in the direction of γ_1), and set $\rho\omega_\mu=iRe^{i\phi}$. Then, as ρ varies over γ_1, ϕ will vary over $0\leqq\phi\leqq\pi/(2n)$, and moreover ϕ will satisfy $0\leqq\phi/2\leqq\sin\phi$. Hence, since $|1-\rho^{4an}/R^{4an}|^m\leqq C\phi^m$, we have

$$\begin{aligned}\left|\int_{\gamma_1}\rho^{k-j}\left(1-\frac{\rho^{4an}}{R^{4an}}\right)^m e^{\rho\omega_\mu x}d\rho\right|\leqq C\int_0^{\pi/(2n)}R^{k-j}\phi^m e^{-R\phi\delta/2}Rd\phi=\frac{C}{R^{m-k+j}}\int_0^{R\pi/(2n)}\xi^m e^{-\xi\delta/2}d\xi\\ <\frac{C}{R^{m-k+j}}\int_0^\infty\xi^m e^{-\xi\delta/2}d\xi<\frac{K}{R^{m-k+j}},\qquad (j=1,2,\cdots,m).\end{aligned}$$

Similar results hold on the other parts of Γ. Hence

$$(10)\qquad \frac{1}{2\pi i}\int_\Gamma\Phi(x,\rho)\left(1-\frac{\rho^{4an}}{R^{4an}}\right)^m d\rho=O\left(\frac{1}{R^{m-k+1}}\right)<O\left(\frac{1}{N^{m-k}}\right).$$

Next, consider the result of multiplying the right hand side of formula (6) by $(1/(2\pi i))(1-\rho^{4an}/R^{4an})^m d\rho$ and integrating over Γ. The integrals which now appear have the form

$$\int_\Gamma \sum_{i=1}^{n} (\rho\omega_i)^s (1-\rho^{4an}/R^{4an})^m d(\rho\omega_i) = \int z^s(1-z^{4an}/R^{4an})^m dz,$$
$$s = k-1, \cdots, k-m,$$

the latter integration being taken over the entire circumference of the circle $|z| = R$. But

$$\int z^s\left(1-\frac{z^{4an}}{R^{4an}}\right)^m dz = \int z^s dz - \frac{m}{R^{4an}}\int z^{4an+s}dz + \cdots + \frac{(-1)^m}{R^{4amn}}\int z^{4amn+s}dz,$$

and the integrals on the right being all either zero or $2\pi i$, it follows that the right hand side may be written

$$\int z^s dz + O\left(\frac{1}{N^m}\right),$$

since, by hypothesis, $4\alpha n \geqq m$. Furthermore, $\int z^s dz = 0$ except when $s = -1$, when it has the value $2\pi i$. Hence, by the argument of section 5, and by virtue of (10), we conclude finally that

$$\sigma_N^{(k)}(x) = f^{(k)}(x) + O\left(\frac{1}{N^{m-k}}\right)$$

uniformly on $0 < \delta \leqq x \leqq 1-\delta$, for $k = 0, 1, \cdots, m-1$.

MOUNT ALLISON UNIVERSITY,
SACKVILLE, N. B., CANADA.

ORDERED TOPOLOGICAL SPACES.*

By Samuel Eilenberg.

1. A topological space[1] X will be called ordered if a relation $<$ is given satisfying the following conditions

(1.1) *For any $x, y \in X$ one and only one of the relations $x < y$, $x = y$, $y < x$ holds,*

(1.2) *If $x, y, z \in X$, $x < y$ and $y < z$, then $x < z$,*

(1.3) *If $x, y \in X$ and $x < y$ then there are neighborhoods $U(x)$ of x and $U(y)$ of y such that $x < y'$ and $x' < y$ whenever $x' \in U(x)$ and $y' \in U(y)$.*

Given $x \in X$ denote by A_x the set of points $y \in X$ such that $y < x$ and by B_x the set of points $y \in X$ such that $x < y$. Conditions (1.1) and (1.3) are then equivalent to the following

(1.1′) $$X - x = A_x + B_x, \qquad A_xB_x = 0,$$

(1.3′) *A_x and B_x are open.*

Although we start with a general topological space we have that

(1.4) *Every ordered space X is a Hausdorff space.*[2]

Proof. Let $x < y$. If there is an element z such that $x < z < y$ then $x \in A_z$ and $y \in B_z$, the sets A_z and B_z being open and disjoint because of (1.1′) and (1.3′).

If no such z exists then $A_yB_x = 0$, $x \in A_y$, $y \in B_x$ and A_y and B_x are open because of (1.3′).

2. Things become simpler if X is supposed connected.

(2.1) *If X is connected then* (1.2) *is a consequence of* (1.1) *and* (1.3).

Proof. Let $x < y$ and $y < z$. Since $z \in B_y$, therefore $X - B_y \subset X - z$ and by (1.1′)

$$A_y + y \subset A_z + B_z.$$

X being connected it follows from (1.1′) and (1.3′) that $A_y + y$ is connected.

---

* Received March 4, 1940.

[1] A topological space is a neighborhood space satisfying the first three axioms of F. Hausdorff, *Grundzuge der Mengenlehre*, Leipzig, 1914, p. 213.

[2] *Ibid.*, p. 213.

Since by (1.1′) and (1.3′), A_z and B_z are open and disjoint and since $y \in A_z$ therefore

$$A_y + y \subset A_z.$$

Since $x \in A_y$ it follows that $x \in A_z$ and therefore $x < z$.

(2.2) *If X is ordered and connected then the order-type of X is continuous.*[3]

Proof. Let $X = P + Q$, $P \neq 0 \neq Q$ be a decomposition such that $x < y$ whenever $x \in P$, $y \in Q$.

Suppose that P has no last element and that Q has no first element. Therefore given $x \in P$ there is a point $x_1 \in P$ such that $x < x_1$. By (1.3) there is a neighborhood $U(x)$ of x such that $x' < x_1$ whenever $x' \in U(x)$. Therefore $U(x) \subset P$ and P is open. Similarly we prove that Q is open. This contradicts the connectedness of X since $PQ = 0$.

Suppose now that x is the last element of P and that y is the first element of Q, then $P = A_y$ and $Q = B_x$. By (1.3′) P and Q are open which leads to a contradiction again.

3. In this section we shall be concerned with the possibility of establishing an order in a given topological connected space X.

Let $X \times X$ be the product space consisting of all couples (x, y) where $x, y \in X$.[4] Let $P(X)$ be the subset of $X \times X$ determined by the condition $x \neq y$.

THEOREM I. *A topological connected*[5] *space X can be ordered if and only if $P(X)$ is not connected.*

Proof. Suppose that X is ordered. Let $A(X)$ be the subset of $P(X)$ consisting of all points (x, y) such that $x < y$. Similarly we define $B(X)$ by the condition $y < x$. From (1.1) it follows that

$$P(X) = A(X) + B(X), \qquad A(X)\, B(X) = 0.$$

From (1.3) it follows that $A(X)$ and $B(X)$ are open. Therefore $P(X)$ is not connected.

Given $(x, y) \in P(X)$ let

$$\Lambda(x, y) = (y, x)$$

Clearly Λ is a homeomorphic transformation of $P(X)$ on itself.

In order to prove the second part of Th. I let us suppose first that a

[3] *Ibid.*, p. 90.

[4] Note that $(x, y) \neq (y, x)$ unless $x = y$.

[5] The case when X contains no more than one point is excluded from our considerations.

decomposition into two open disjoint sets $P(X) = A + B$ is given such that $\Lambda(A) = B$. Define the relation $<$ by writing $x < y$ if and only if $(x, y) \in A$. Clearly (1.1) and (1.3) are satisfied. Since X is connected it follows from (2.1) that (1.2) is satisfied too and therefore X is ordered by the relation $<$. Hence the proof of Th. I reduces to the following

(3.1) *If X is connected and $P(X)$ is not connected, then $P(X)$ consists of two components A and B such that $\Lambda(A) = B$.*

Proof. Assume on the contrary that

$$P(X) = C_1 + C_2, \quad C_1 \neq 0 \neq C_2, \quad C_1\bar{C}_2 = 0 = C_2\bar{C}_1, \quad C_1\Lambda(C_1) \neq 0$$

Let $D_1 = C_1\Lambda(C_1)$ and $D_2 = C_2 + \Lambda(C_2)$. We then have

(*) $\quad P(X) = D_1 + D_2, \quad D_1 \neq 0 \neq D_2, \quad D_1\bar{D}_2 = 0 = D_2\bar{D}_1,$

(**) $\quad \Lambda(D_1) = D_1, \quad \Lambda(D_2) = D_2.$

Given $x \in X$ let M_x be the set of points y such that $(x, y) \in D_1$ and let N_x be the set of points y such that $(x, y) \in D_2$. It follows from (*) that

$$X - x = M_x + N_x, \qquad M_x\bar{N}_x = 0 = \bar{M}_xN_x.$$

X being connected this implies that the sets $\bar{M}_x = M_x + x$ and $\bar{N}_x = N_x + x$ are connected.

Let $y \in M_x$ and $z \in N_x$. It follows that $(x, y) \in D_1$ and y *non* $\in N_x$. Therefore $\bar{N}_x \times y \subset P(X) = D_1 + D_2$. Since $N_x \times y$ is connected and $(x, y) \in N_x \times y$ therefore $\bar{N}_x \times y \subset D_1$ and in particular $(z, y) \in D_1$. Similarly we have $\bar{M}_x \times z \subset D_2$ and therefore $(y, z) \in D_2$. This, however, contradicts (**) and therefore one of the sets M_x or N_x must be empty.

Let $(x, y) \in D_1$. It follows that $M_x \neq 0$, therefore $M_x = X - x$ and $(x, y') \in D_1$ for every $y' \in X - x$. By (**) we have then also $(y', x) \in D_1$. Consequently $M_{y'} \neq 0$ and therefore $M_{y'} = X - y'$ for every $y' \in X - x$. We have therefore proved that $M_x = X - x$ for every $x \in X$, therefore $D_1 = P(X)$ and $D_2 = 0$ contradicting (*).

Comparing (3.1) with the argument used above to prove that $P(X)$ is not connected if X is connected and ordered we obtain that

(3.2) *If X is ordered and connected*[5] *then $A(X)$ and $B(X)$ are the components of $P(X)$.*

4. In this section the uniqueness of the order in a connected space will be established.

(4.1) *Let X and Y be two ordered connected spaces. Every* (1-1) *continuous mapping of X on Y either preserves the order or reverses it.*

Proof. Let ϕ be the mapping of X on Y. Given $(x, y) \epsilon P(X)$ let

$$\psi(x, y) = (\phi(x), \phi(y))$$

Clearly ψ is a (1-1) continuous mapping of $P(X)$ on $P(Y)$. Because of (3.2) we have either $\psi(A(X)) = A(Y)$ or $\psi(A(X)) = B(Y)$. In the first case ϕ preserves the order, in the other it reverses it.

THEOREM II. *Two orderings of a connected topological space X are either identical or inverse to each other.*

Proof, follows from (4.1) by taking $X = Y$ and considering the identity transformation of X on itself.

5. In this section we shall establish an inverse of (4.1) under the additional hypothesis of local connectedness.

(5.1) *Let X be an ordered space, Y an ordered connected space, and A an open and connected subset of Y. For every* (1-1) *order preserving* (*or reversing*) *mapping ϕ of X on Y the set $\phi^{-1}(A)$ is open.*

Proof. By (2.2) the order-type of Y is continuous. Since A is connected it follows that $Y = P + A + Q$ where $y' \epsilon P$, $y \epsilon A$ and $y'' \epsilon Q$ imply $y' < y < y''$. Since A is open it can easily be seen that if $P \neq 0$ then there is a last element y_1 of P. Similarly if $Q \neq 0$ there is a first element y_2 of Q. It follows that either $A = Y$, or $A = A_{y_2}$, or $A = B_{y_1}$, or $A = A_{y_2}B_{x_1}$.

Let $x_1 = \phi^{-1}(y_1)$ and $x_2 = \phi^{-1}(y_2)$. Since ϕ is order preserving therefore $\phi^{-1}(A_{y_2}) = A_{x_2}$ and $\phi^{-1}(B_{y_1}) = B_{x_1}$. It follows that either $\phi^{-1}(A) = X$, or $\phi^{-1}(A) = A_{x_2}$, or $\phi^{-1}(A) = B_{x_1}$ or $\phi^{-1}(A) = A_{x_2}B_{x_1}$. Hence $\phi^{-1}(A)$ is open by (1.3′).

(5.2) *Let X be an ordered space and Y an ordered connected and locally connected space. Every* (1-1) *order preserving* (*or reversing*) *mapping of X on Y is continuous.*

Proof. Let ϕ be the mapping. From (5.1) it follows that $\phi^{-1}(A)$ is open for every open connected set $A \subset Y$. Since Y is locally connected this implies that $\phi^{-1}(U)$ is open for every open set $U \subset Y$. Therefore ϕ is continuous.

6. Adding separability to our hypotheses we shall give a new characterization of the connected subsets of the linear continuum.

(6.1) *A connected separable topological space X can be mapped* (1-1) *and continuously on a subset of the linear continuum if and only if $P(X)$ is not connected.*[6]

Proof. Let ϕ be a (1-1) continuous mapping of X on a linear set Y. Clearly $P(Y)$ is then a (1-1) continuous image of $P(X)$ and since $P(Y)$ is not connected, $P(X)$ is not connected.

Suppose now that $P(X)$ is not connected. By Th. I the space X can be ordered and by (2.2) its order type will be continuous.

Let $A \subset X$ be an enumerable set such that $\bar{A} = X$. Let $x < y$; since the order of X is continuous there is a z such that $x < z < y$. Therefore $A_y B_x \neq 0$ and since $A_y B_x$ is open by (1.3′) it follows that $A A_y B_x \neq 0$ which means that there is a $z' \epsilon A$ such that $x < z' < y$. The set A is therefore dense in X in the sense of order.

Since the order type of X is continuous and there is an enumerable subset of X dense in X in the sense of order, therefore there is a (1-1) order preserving correspondence ϕ mapping X on the open, half-open or closed interval Y on the straight line.[7] Since Y is locally connected, therefore by (5.2) ϕ is continuous. This proves (6.1).

The inverse ϕ^{-1} is also (1-1) and order preserving. Hence if X is locally connected ϕ^{-1} is continuous by (5.2) and ϕ is a homeomorphism. Hence we have proved

THEOREM III. *A connected*[5] *locally connected separable topological space X is homeomorphic with a subset of the linear continuum if and only if $P(X)$ is not connected.*

7. Let X_α and X_β be two topologizations of an abstract set X. The topology of X_α is said to be *stronger* than the topology of X_β if the class of open sets in X_α is a proper subclass of the class of open sets in X_β, in other words if the identity mapping of X on itself induces a continuous mapping of X_β on X_α which is not a homeomorphism. In this way all possible topologizations of X form a partially ordered set.[8]

Given an ordered abstract set X we shall consider the class $[X]$ of all topologizations of X which lead to an ordered topological space with respect to the given order. It follows from (1.4) that $[X]$ contains only Hausdorff topologies.

---

[6] For X compact this theorem was announced without proof by C. Pauc, C. R. Paris 203 (1936), p. 154.

[7] Hausdorff, *loc. cit.*, p. 101.

[8] Garrett Birkhoff, *Fundamenta Mathematicae*, vol. 26 (1936), p. 156.

We may consider X with the discrete topology X_0, i. e. the topology for which every set is open. Of course X_0 is the weakest topology in the class $[X]$.

We shall now define a topology X_1 for X which will be the strongest one in the class $[X]$.

Let $x, y \epsilon X$ and $x < y$. Consider every one of the sets A_x, $C_{xy} = A_y B_x$, B_y as a neighborhood of every one of its points. Let X_1 be the topological space thus obtained. It is clear that (1. 3′) is satisfied and therefore X_1 belongs to the class $[X]$.

Given any topology X_a in $[X]$ the sets A_x, C_{xy} and B_y are open because of (1. 3′). It follows that X_1 is not weaker than X_a. Conversely given any topology X_a not weaker than X_1 the sets A_x and B_x are open in X. (1. 3′) being satisfied X_a belongs to $[X]$. Hence we have proved that

(7. 1) *The class* $[X]$ *consists of* X_1 *and all topologies weaker than* X_1.

Since the topology X_1 is entirely defined in terms of order, therefore

(7. 2) *If* X *and* Y *are ordered sets, then every* (1-1) *order preserving mapping of* X *on* Y *induces an order preserving homeomorphism of* X_1 *and* Y_1.

8. In this section the structure of X_1 will be described in the case when X is ordered continuously.

(8. 1) *For every ordered set* X *the following conditions are equivalent:*

(a) *The order-type of* X *is continuous,*

(b) X_1 *is connected and locally connected,*

(c) X *can be topologized so as to become an ordered connected topological space.*

Proof. Clearly (b) implies (c). From (2. 2) it follows that (c) implies (a). We shall prove now that (a) implies (b).

Let $x < y$ and let Y be one of the sets X, A_x, $A_y B_x$ or A_y. We shall prove that if (a) holds then Y is a connected subset of X_1. Assume on the contrary that

$$(*) \qquad Y = D_1 + D_2, \qquad D_1 \bar{D}_2 = 0 = \bar{D}_1 D_2, \qquad D_1 \neq 0 \neq D_2.$$

Let $x_1 \epsilon D_1$, $x_2 \epsilon D_2$, $x_1 < x_2$ and let

$$P = D_1 A_{x_2} + \Sigma A_z \text{ where } z \epsilon D_1 A_{x_2}, \qquad Q = X - P.$$

Clearly $x_1 \epsilon P$ and $x_2 \epsilon Q$. Also if $x' \epsilon P$ and $x'' < x'$ then $x'' \epsilon P$. It follows that

$$X = P + Q, \qquad P \neq 0 \neq Q, \qquad PQ = 0,$$

and that $x < y$ whenever $x \in P$, $y \in Q$. The ordering of X being continuous it follows that either P has a last element or Q a first element.

Suppose that P has a last element x_0. It follows that $x_0 \in D_1 A_{x_2}$ and that $z \in D_2$ for every element $z \in Y$ such that $x_0 < z < x_2$, or in other words that $YC_{x_0 x_2} \subset D_2$. A glance at the definition of Y shows that $C_{x_0 x_2} \subset Y$ since $x_0, x_2 \in Y$. It follows that $C_{x_0 x_2} \subset D_2$. Since the ordering of X is continuous it is clear that the set $C_{x_0 x_2}$ has points in common with every neighborhood of x_0 and therefore $x_0 \in \bar{D}_2$. It follows that $x_0 \in D_1 \bar{D}_2$ contradicting (*).

Suppose now that Q has a first element y_0. Since $x_2 \in D_2 Q$ therefore either $y_0 = x_2$ or $y_0 < x_2$. In both cases it follows from the definition of P that $y_0 \in D_2$.

Given any $y < y_0$ we have $C_{y y_0} \subset P$ and therefore $D_1 C_{y y_0} \neq 0$. Since the ordering of X is continuous every neighborhood U of y_0 must contain the set $C_{y y_0}$ for some $y < y_0$. It follows that $D_1 U \neq 0$, therefore $y_0 \in \bar{D}_1$ and finally $y_0 \in \bar{D}_1 D_2$ contradicting (*)

(8.2) *Given a set X ordered continuously, there is exactly one topologization of X which leads to a connected and locally connected ordered space.*

Proof. It follows from (8.1) that there is at least one such topologization. Let X_α and X_β be two topologizations of X both leading to connected and locally connected ordered spaces. The identity transformation of X on itself induces an order preserving (1-1) transformation of X_α on X_β and conversely. Since X_α and X_β are both connected and locally connected, it follows from (5.2) that these transformations are continuous and therefore establish the identity of X_α and X_β.

UNIVERSITY OF MICHIGAN.

A GENERALIZATION OF A METRIC SPACE WITH APPLICATIONS TO SPACES WHOSE ELEMENTS ARE SETS.*

By G. Baley Price.

1. Introduction. The metric spaces were among the earliest of the abstract spaces studied; later the general topological spaces were developed. This paper shows how a topological structure can be introduced by means of a function which is more general than the ordinary distance function and reduces to it in a special case, and investigates the properties of this topology. Conditions under which the space is metricisable are given in section 5. The investigation was suggested by a study of topological structures in spaces whose elements are sets (see Hausdorff [7, pp. 145-150],[1] Kuratowski [9, pp. 89-92, 152-158]) and, in particular, by structures based on symmetric differences of sets (see MacNeille [10], Wazewski [17], Fréchet [4], Nikodym [13], Szpilrajn [15]). The paper is divided into two parts; the theory of the generalized metric space is developed in the first part, and the results are applied to spaces whose elements are sets in the second part. The values of the generalized distance function are elements in a certain partially ordered space; the results thus have some connection, although not close, with Kantorovitch's [8] partially ordered spaces.

PART I. A Generalization of a Metric Space.

2. The partially ordered space $\mathfrak{D}$. Let $\mathfrak{D}$ be a space with elements d, $e, \cdots$, a partial order $<$ defined for some pairs of elements in $\mathfrak{D}$, and an operation $+$ defined for every pair of elements. We make the following postulates concerning $\mathfrak{D}$:

(2.1) The relation $d < e$ holds for some pairs of elements d, e in $\mathfrak{D}$.

(2.2) The partial ordering is transitive, that is, $(d < e)\ (e < f) \rightarrow (d < f)$.

(2.3) $(d + e) \epsilon \mathfrak{D}$ and $d + e = e + d$ for every pair of elements d, e in $\mathfrak{D}$.

(2.4) $(d_1 < e_1)\ (d_2 \leqq e_2) \rightarrow (d_1 + d_2 < e_1 + e_2)$.

3. The metric. Let $\mathfrak{E}$ be a subset of $\mathfrak{D}$ with the following property:

* Received July 18, 1939; Revised August 15, 1940.

[1] Numbers in square brackets refer to the bibliography at the end.

(3.1) Hypothesis I. *Given any element e_0 in $\mathfrak{E}$, there exist elements e_1, e_2 in $\mathfrak{E}$ such that $e_1 + e_2 \leqq e_0$.*

Let K be a space with elements $x, y, \cdots$. Let $d(x,y)$ be a function with values in $\mathfrak{D}$ which is defined for every pair of elements x, y in K, and which has the following properties:

(3.2) $d(x,x) < e$ for every e in $\mathfrak{E}$ and x in K;

(3.3) $d(x,y) < e$ for every e in $\mathfrak{E}$ implies $x = y$;

(3.4) $d(x,y) = d(y,x)$;

(3.5) $d(x,z) \leqq d(x,y) + d(y,z)$.

4. Topological structure in K. The set of points x in K such that $d(x,x_0) < e_0$, where $x_0 \in K$ and $e_0 \in \mathfrak{E}$, will be called a sphere with center x_0 and radius e_0. A point x_0 is a point of accumulation of a set $E \subseteqq K$ if and only if every sphere with center x_0 contains a point of E distinct from x_0. A set is closed if it contains all of its points of accumulation and open if it is the complement of a closed set. The derived set E' of E is defined in the usual way.

(4.1) Theorem. *If x_1, x_2 are two distinct elements of K, there exist two disjoint spheres with centers x_1, x_2.*

By (3.3) there exists a sphere $d(x,x_1) < e_0$ which does not contain x_2. Let e_1, e_2 be elements of $\mathfrak{E}$ such that $e_1 + e_2 \leqq e_0$. If the spheres $d(x,x_1) < e_1$, $d(x,x_2) < e_2$ had an element x in common, we would have $d(x_1,x_2) \leqq d(x_1,x) + d(x,x_2) < e_1 + e_2 \leqq e_0$, a contradiction of the definition of e_0.

5. Further hypotheses on $\mathfrak{E}$ and their consequences. In this section we shall assume without further mention that $\mathfrak{D}$ and $\mathfrak{E}$ satisfy all of the assumptions of sections 2 and 3, and that $\mathfrak{E}$ satisfies certain further hypotheses as stated.

(5.1) Hypothesis II. *Given any two elements $e_0 \in \mathfrak{E}$ and $d \in \mathfrak{D}$, $d < e_0$, there exists an element $e \in \mathfrak{E}$ such that $d + e \leqq e_0$.*

(5.2) Theorem. *If $\mathfrak{E}$ satisfies Hypothesis II, every sphere $d(x,x_0) < e_0$ is an open set.*

Let x_1 be any point of $d(x,x_0) < e_0$. Then $d(x_1,x_0) = d < e_0$. By Hypothesis II there is an e in $\mathfrak{E}$ such that the sphere $d(x,x_1) < e$ is contained in $d(x_1,x_0) < e_0$, for $d(x,x_0) \leqq d(x,x_1) + d(x_1,x_0) < d + e \leqq e_0$ and

$d(x, x_0) < e_0$ by (2.2). Since every point of $d(x, x_0) < e_0$ is the center of some sphere contained in the set, it is an open set.

(5.3) HYPOTHESIS III. *Given any two elements e_1, e_2 in $\mathfrak{E}$, there exists a third element e in $\mathfrak{E}$ such that $e \leqq e_1$ and $e \leqq e_2$.*

From Hypotheses I and III it follows that for each e_0 in $\mathfrak{E}$ there exists an element e such that $e + e \leqq e_0$. For there exist elements e_1, e_2 such that $e_1 + e_2 \leqq e_0$, and an element e such that $e \leqq e_1$ and $e \leqq e_2$. Then $e + e \leqq e_0$ by (2.2) and (2.4). The following theorem is a consequence of this fact.

(5.4) THEOREM. *If $\mathfrak{E}$ satisfies Hypothesis III, there is a sphere $d(x, x_0) < e$ contained in the product of any two spheres $d(x, x_0) < e_1$, $d(x, x_0) < e_2$.*

(5.5) THEOREM. *If $\mathfrak{E}$ satisfies Hypotheses I, II, III, the space K is regular, that is, if G is an open set, and if $x_0 \in G$, there exists an open set which contains x and whose closure is contained in G.*

Since G is open, there is a sphere $d(x, x_0) < e_0$ contained in G. By Hypotheses I, III there is an element e such that $e + e \leqq e_0$. We shall show that the sphere $d(x, x_0) < e$, which is open by Theorem 5.2, is a set whose closure is contained in $d(x, x_0) < e_0$ and hence in G. Suppose a point x^* not contained in $d(x, x_0) < e_0$ were a point of accumulation of the set $d(x, x_0) < e$. Then the sphere $d(x, x^*) < e$ contains an element x_1 in $d(x, x_0) < e$, and $d(x^*, x_0) \leqq d(x^*, x_1) + d(x_1, x_0) < e + e \leqq e_0$. But $d(x^*, x_0) < e_0$ contradicts the assumption that x^* is not contained in $d(x, x_0) < e_0$. The proof is complete.

(5.6) THEOREM. *If $\mathfrak{E}$ satisfies Hypotheses I, II, III, and if a sphere with center x_0 be defined as a neighborhood of x_0, then K is a regular Hausdorff space.*

The first postulate of Hausdorff (see Kuratowski [9, p. 28]) is satisfied since every point x in K has neighborhoods. The next postulate is satisfied as a result of Theorem 5.4. If $d(x, x_0) < e_0$ is a neighborhood of x_0 and x_1 is a point in it, there is a neighborhood of x_1 contained in the given neighborhood of x_0; this result follows from Hypothesis II as shown in the proof of Theorem 5.2. Again, if x_1 and x_2 are two distinct elements of K, there exist, by Theorem 4.1, neighborhoods $d(x, x_1) < e_1$ and $d(x, x_2) < e_2$ without common points. Finally, the space is regular by Theorem 5.5.

A set E in K is said to be totally bounded if it can be covered by a finite number of spheres of any given radius e in $\mathfrak{E}$. A set E is said to be compact in K if every infinite set in E has a point of accumulation in K.

(5.7) THEOREM. *If $\mathfrak{E}$ satisfies Hypothesis III, a compact set E is totally bounded.*

Let e in $\mathfrak{E}$ be given. Choose any two elements x_1, x_2 in E such that x_2 is not contained in $d(x, x_1) < e$. Choose a third point x_3 which is not contained in either of the spheres $d(x, x_1) < e$, $d(x, x_2) < e$. Continue in this manner to choose points $x_1, x_2, \cdots$; if this set is infinite, it has a point of accumulation x_0. By Hypothesis I there exist elements e_1, e'_2 in $\mathfrak{E}$ such that $e_1 + e'_2 \leqq e$. The sphere $d(x, x_0) < e_1$ contains a point x_m, distinct from x_0, of the set selected. By Theorem 4.1 there is a sphere $d(x, x_0) < e''_2$ which does not contain x_m. By Hypothesis III there is an element e_2 in $\mathfrak{E}$ such that $e_2 \leqq e'_2$, $e_2 \leqq e''_2$. The sphere $d(x, x_0) < e_2$ contains an element x_n distinct from x_0; since $d(x, x_0) < e_2$ is contained in $d(x, x_0) < e''_2$, which does not contain x_m, the elements x_m and x_n are distinct. But $d(x_m, x_n) \leqq d(x_m, x_0) + d(x_0, x_n) < e_1 + e_2 \leqq e_1 + e'_2 \leqq e$. This contradiction of the definition of the set $x_1, x_2, \cdots$ establishes the theorem.

(5.8) HYPOTHESIS IV. *Given any element d in $\mathfrak{D}$, there is an element e in $\mathfrak{E}$ such that $d \leqq e$.*

A set E in K is said to be bounded if and only if there is an element e in $\mathfrak{E}$ such that $d(x, y) < e$ for every pair of elements x, y in E. If $\mathfrak{E}$ satisfies Hypothesis IV, the sum of any two bounded sets is a bounded set. Let $d(x, x_0) < e_1$ for x, x_0 in E_1 and $d(y, y_0) < e_2$ for y, y_0 in E_2. Then $d(x, y) \leqq d(x, x_0) + d(x_0, y_0) + d(y_0, y)$, and the stated result follows from (2.3), (2.4), and Hypothesis IV.

(5.9) HYPOTHESIS V. *Given any two elements e_1, e_2 of $\mathfrak{E}$, there is an element e in $\mathfrak{E}$ such that $e_1 \leqq e$ and $e_2 \leqq e$.*

If $\mathfrak{E}$ satisfies Hypotheses IV and V, a totally bounded set E is bounded. For since E is totally bounded, it can be covered by a finite number of spheres $d(x, x_1) < e_0, \cdots, d(x, x_n) < e_0$. Let y_i denote any point in $d(x, x_i) < e_0$. From (3.5), (2.3), and Hypothesis IV it follows that $d(y_i, y_j) < e_0 + d(x_i, x_j) + e_0 \leqq e_{ij}$, where $e_{ij} \in \mathfrak{E}$. By (2.2) and a repeated application of Hypothesis V we see that there exists an element e in $\mathfrak{E}$ such that $e_{ij} \leqq e$ for all i and j. It follows that E is bounded.

(5.10) HYPOTHESIS VI. *There exists a denumerable set of elements $e_1, e_2, e_3, \cdots$ in $\mathfrak{E}$, to be denoted by $\{e_k\}$, such that if e is any element in $\mathfrak{E}$, there is an element e_k for which $e_k \leqq e$.*

(5.11) THEOREM. *If $\mathfrak{E}$ satisfies Hypotheses I, III, VI, the space K is metricisable.*

First we shall show that $\{e_k\}$ contains a subset $\{e_{k_i}\}$ with the following properties:

(5.12) $e_{k_1} \geqq e_{k_2} \geqq \cdots \geqq e_{k_i} \geqq \cdots$;

(5.13) $e_{k_i} + e_{k_i} \leqq e_{k_{i-1}}$;

(5.14) given any e_k in $\{e_k\}$, there exists an e_{k_i} in $\{e_{k_i}\}$ such that $e_{k_i} \leqq e_k$.

It follows from Hypothesis VI and (5.14) that the points of accumulation of sets and the limits of sequences are the same in terms of $\mathfrak{E}$, $\{e_k\}$, and $\{e_{k_i}\}$.

The subset $\{e_{k_i}\}$ of $\{e_k\}$ can be chosen as follows. Set $e_{k_1} = e_1$. Let e^*_1 be an element in $\mathfrak{E}$ such that $e^*_1 + e^*_1 \leqq e_{k_1}$; such an element exists by Hypotheses I and III. Choose e_{k_2} as an element of $\{e_k\}$ such that $e_{k_2} \leqq e^*_1$, e_1, e_2. This choice is possible, for by Hypothesis III and (2.2) there is an element e'_1 of $\mathfrak{E}$ such that $e'_1 \leqq e^*_1$, e_1, e_2; by Hypothesis VI there is an element e_{k_2} in $\{e_k\}$ such that $e_{k_2} \leqq e'_1$; finally, by (2.2) and (2.4), $e_{k_2} \leqq e^*_1$, e_1, e_2 and $e_{k_2} + e_{k_2} \leqq e^*_1 + e^*_1 \leqq e_{k_1}$. This process can be repeated, for suppose that elements $e_{k_1}, \cdots, e_{k_r}$ satisfying (5.12), (5.13) have been chosen. Choose elements e^*_r, e'_r in $\mathfrak{E}$ such that $e^*_r + e^*_r \leqq e_{k_r}$ and $e'_r \leqq e^*_r$, e_{k_r}, e_1, $e_2, \cdots, e_{r+1}$. Finally, choose $e_{k_{r+1}}$ as an element of $\{e_k\}$ such that $e_{k_{r+1}} \leqq e'_r$. It is clear that the set thus chosen satisfies (5.12), (5.13), (5.14).

In terms of the sequence (5.12) we shall define a distance function $\rho(x, y)$ with the following properties:

(5.15) for each pair of elements x, y in K, $\rho(x, y)$ is a non-negative real number;

(5.16) $\rho(x, y) = 0$ if and only if $x = y$;

(5.17) $\rho(x, y) = \rho(y, x)$;

(5.18) if $\rho(x, y) < \epsilon$ and $\rho(y, z) < \epsilon$, then $\rho(x, z) < 2\epsilon$.

Such a function can be defined as follows: set $\rho(x, y) = 1/2^i$ if and only if $d(x, y) < e_{k_i}$ is true and $d(x, y) < e_{k_{i+1}}$ is false; set $\rho(x, y) = 0$ if and only if $d(x, y) < e_{k_i}$ for all i. Then from the properties of $d(x, y)$, it is obvious that $\rho(x, y)$ satisfies (5.15), (5.16), (5.17). Also it satisfies (5.18), for suppose $\rho(x, y) = 1/2^i$ and $\rho(y, z) = 1/2^j \leqq 1/2^i$. Then $d(x, z) \leqq d(x, y) + d(y, z) < e_{k_i} + e_{k_j} \leqq e_{k_i} + e_{k_i} \leqq e_{k_{i-1}}$, and $\rho(x, z) \leqq 1/2^{i-1}$. It follows easily that (5.18) is satisfied. It is to be observed that the topology based on the distance function $\rho(x, y)$ is equivalent to that based on $d(x, y)$ and the set $\{e_k\}$.

Finally, Frink [5] has shown that it is possible to metricise K by a distance function which is topologically equivalent to $\rho(x, y)$. The proof is complete.

6. First examples. Let $\mathfrak{D}$ be the class of real numbers $x \geqq 0$, and let $\mathfrak{E}$ be the real numbers $x > 0$. Furthermore, let $<$ and $+$ have their usual meaning in the system of real numbers. Then $\mathfrak{D}$ satisfies all the assumptions in section 2, and $\mathfrak{E}$ satisfies Hypotheses I-VI. A space K with a metric $d(x, y)$ having the properties stated in section 3 is an ordinary metric space.

Consider next the partially ordered spaces of Kantorovitch [8]. Let X be a class of elements x which form an additive abelian group. Furthermore, let there be a relation $>$ defined so that for some of the elements x in X the relation $x > 0$ holds. Kantorovitch assumes that this system satisfies the following postulates:

(6.1) The relation $x > 0$ excludes $x = 0$.

(6.2) If $x_1 > 0$ and $x_2 > 0$, then $x_1 + x_2 > 0$.

(6.3) To each element $x \in X$ there corresponds at least one element $x_1 \in X$ such that $x_1 \geqq 0$ and $x_1 - x \geqq 0$.

(6.4) For every set E bounded above there exists a least upper bound sup E.

In some cases it is assumed in addition that X is a vector space over the real number system. Then the following postulate is applicable:
(6.5) If $x > 0$, and if $\lambda > 0$ is a real number, then $\lambda x > 0$.

If the first four postulates are satisfied in X, it is called a partially ordered topological group. If in addition the fifth postulate is satisfied in X, it is called a linear partially ordered space.

If $x_2 - x_1 > 0$, we say $x_2 > x_1$. In a partially ordered space in which the first four postulates are satisfied it is possible to define an absolute value $|x|$ of x; the absolute value of x is an element in X and has the formal properties of the absolute value of a real number.

Let X be a linear partially ordered space, $\mathfrak{D}$ the subset of elements $x \geqq 0$, and $\mathfrak{E}$ the subset of elements $x > 0$. Then $\mathfrak{D}$ and $\mathfrak{E}$ satisfy all the assumptions of sections 2 and 3. If K be taken as X, and if $d(x, y)$ be defined as $|x - y|$, it follows from the properties of the absolute value in X that $d(x, y)$ has properties (3.2), (3.3), (3.4), (3.5). It should be observed, however, that $\mathfrak{E}$ may not satisfy Hypothesis III in this case, in fact, a simple linear partially ordered space can be formed from the Euclidean plane in which $\mathfrak{E}$ satisfies Hypotheses I, II, IV, V, VI but fails to satisfy Hypothesis III. The sum $x_1 + x_2$ of two elements x_1: (ξ_1, η_1) and x_2: (ξ_2, η_2) is the element $(\xi_1 + \xi_2, \eta_1 + \eta_2)$; the element x: (ξ, η) follows 0: $(0, 0)$, that is $x > 0$, if and only if $\xi \geqq 0$ and $\eta \geqq 0$ with the inequality holding in at least one case; the absolute value $|x|$ of x: (ξ, η) is $(|\xi|, |\eta|)$. The only element which pre-

cedes two elements $(\xi, 0)$ and $(0, \eta)$ in $\mathfrak{D}$ is $(0, 0)$, which is not in $\mathfrak{E}$; hence Hypothesis III is not satisfied. It should be observed also that there are many other subsets of $\mathfrak{D}$ which satisfy Hypothesis I as well as certain of the others and may therefore be chosen as the set $\mathfrak{E}$. Finally, it may be observed that (6.5) is a much stronger hypothesis than is required for Hypotheses III and V.

Part II. Spaces Whose Elements Are Sets.

7. Symmetric differences of sets. Let K denote a class of elements with subsets $X, Y, \cdots$. The symmetric difference $s(X_1, X_2)$ of two sets X_1, X_2 is defined by

$$(7.1)\quad s(X_1, X_2) = X_1 + X_2 - X_1X_2.$$

The symmetric differences of sets have the following properties:

$$(7.2)\quad \begin{aligned} &\text{(a)}\quad s(X, X) = 0; \\ &\text{(b)}\quad s(X_1, X_2) = s(X_2, X_1); \\ &\text{(c)}\quad s(X_1, X_2) \supset 0 \qquad X_1 \neq X_2; \\ &\text{(d)}\quad s(X_1, X_3) \subseteqq s(X_1, X_2) + s(X_2, X_3). \end{aligned}$$

$$(7.3)\quad s(X_1, X_1 + X_2) = s(X_1X_2, X_2).$$

$$(7.4)\quad s(X_1 + X_2 + \cdots, Y_1 + Y_2 + \cdots) \subseteqq s(X_1, Y_1) + s(X_2, Y_2) + \cdots.$$

$$(7.5)\quad s(X_1X_2\cdots, Y_1Y_2\cdots) \subseteqq s(X_1, Y_1) + s(X_2, Y_2) + \cdots.$$

(7.6) For every $m \geqq 1$

$$\begin{aligned} &\text{(a)}\quad s[X, (X_1X_2\cdots) + (X_2X_3\cdots) + \cdots] \subseteqq s(X, X_m) + s(X, X_{m+1}) + \cdots; \\ &\text{(b)}\quad s[X, (X_1 + X_2 + \cdots)(X_2 + X_3 + \cdots)\cdots] \subseteqq s(X, X_m) + s(X, X_{m+1}) + \cdots. \end{aligned}$$

(7.7) For every $n \geqq 1$

$$\begin{aligned} &\text{(a)}\quad s[X, (X_1X_2\cdots) + (X_2X_3\cdots) + \cdots] \subseteqq s(X, X_{n+1}) + s(X_{n+1}, X_{n+2}) + \cdots; \\ &\text{(b)}\quad s[X, (X_1 + X_2 + \cdots)(X_2 + X_3 + \cdots)\cdots] \subseteqq s(X, X_{n+1}) + s(X_{n+1}, X_{n+2}) + \cdots. \end{aligned}$$

$$(7.8)\quad [(X_1 + X_3 \supseteqq X_2)(X_1X_3 \subseteqq X_2)] \equiv [s(X_1, X_3) = s(X_1, X_2) + s(X_2, X_3)] \equiv [s(X_1, X_2)s(X_2, X_3) = 0].$$

$$(7.9)\quad \begin{aligned} &\text{(a)}\quad s(X_1, X_2) = s(X_1, X_1X_2) + s(X_1X_2, X_2); \\ &\text{(b)}\quad s(X_1, X_2) = s(X_1, X_1X_2) + s(X_1, X_1 + X_2). \end{aligned}$$

The first relation in (7.9) follows from (7.8); the second follows from

the first and (7.3). The verification of the remaining relations can be supplied from (7.1).

8. Introduction of a structure in a system of sets $\mathfrak{M}$. Let $\mathfrak{M}$ denote a system of sets in K which is a ring, a field, a σ-system, and a δ-system (see Hahn [6, pp. 10-20]). The symmetric difference of two sets in $\mathfrak{M}$ is a set in $\mathfrak{M}$. Let the sets X in $\mathfrak{M}$ be the elements of the space $\mathfrak{D}$ (see section 2). We introduce a partial order in $\mathfrak{D}$ in terms of point-set inclusion; more precisely, if $X, Y \epsilon \mathfrak{D}$, $X < Y$ is defined to be equivalent to $X \subseteqq Y$. Finally, the operation $+$ of section 2 is here defined to be point-set addition of sets in $\mathfrak{D}$. It can be verified that (2.1)-(2.4) hold for the partial order $<$ and the operation $+$ as thus defined.

Next, let $\mathfrak{E}$ with elements E, denote a subset of $\mathfrak{M}$ which does not contain the empty set, but which is such that the product of the sets in $\mathfrak{E}$ is the null set. We observe that Hypotheses I and II (see (3.1), (5.1)) are automatically satisfied in $\mathfrak{E}$ since $E_0 + E_0 \leqq E_0$ and $D + E_0 \leqq E_0$ if $D < E_0$.

Finally, set $d(X, Y) = s(X, Y)$, the symmetric difference of X and Y. Then $d(X, Y)$ is defined for every pair of elements X, Y in $\mathfrak{M}$, has its values in $\mathfrak{D}$, and has the properties (3.2), (3.3), (3.4), (3.5) by (7.2). The results in sections 4 and 5 can therefore be applied in $\mathfrak{M}$.

(8.1) THEOREM. *$\mathfrak{M}$ is a complete space.*

Let $X_1, X_2, \cdots$ be a Cauchy sequence; then for every E in $\mathfrak{E}$ there exists an $n(E)$ such that $s(X_m, X_n) \subseteqq E$, that is, $d(X_m, X_n) < E$, for $m > n \geqq n(E)$. Then from (7.7) the sequence has the limit $(X_1X_2 \cdots) + (X_2X_3 \cdots) + \cdots$ or $(X_1 + X_2 + \cdots)(X_2 + X_3 + \cdots) \cdots$, these two elements being equal. Since $\mathfrak{M}$ is a σ-system and a δ-system, the limit of the sequence is an element in $\mathfrak{M}$.

If $\mathfrak{E}$ is a sequence of sets $E_1 \supset E_2 \supset E_3 \supset \cdots$ whose product is the null set, Hypotheses III and VI are satisfied. Then by Theorem 5.11 there is a metric space M which is equivalent to $\mathfrak{M}$. A somewhat stronger result can be established directly. If X_1, X_2 are two elements in $\mathfrak{M}$ such that $s(X_1, X_2)$ is not contained in E_1, we set $D(X_1, X_2) = 1$, where $D(X_1, X_2)$ is the distance between the elements X_1, X_2 of M. In general, if $s(X_1, X_2)$ is contained in E_i but not in E_{i+1}, we set $D(X_1, X_2) = 1/2^i$; if $s(X_1, X_2) \subseteqq E_i$ for all i, we set $D(X_1, X_2) = 0$. Two elements X_1, X_2 are defined to be equal in M if and only if $D(X_1, X_2) = 0$. Then the distance between two elements of M is positive or zero and symmetric; equality in M is equivalent to equality in $\mathfrak{M}$; and the triangle inequality holds in the stronger form $D(X_1, X_3) \leqq \max [D(X_1, X_2), D(X_2, X_3)]$. In a sphere about any point in either space

there is a sphere of the other space and with the same center; hence, points of accumulation are the same in $\mathfrak{M}$ and M.

9. A metric space. In this section let $\mathfrak{M}$ denote the system of all subsets of the unit interval $0 \leqq x \leqq 1$. From (7.2) and the properties of exterior measure it follows that $\mathfrak{M}$ becomes a metric space M if the distance $D(X_1, X_2)$ between two elements X_1, X_2 in $\mathfrak{M}$ is defined to be $m_e[s(X_1, X_2)]$, where $m_e[A]$ denotes the exterior measure of A. Two sets are equal in M if their symmetric difference is a set of measure zero.

A sequence $X_1, X_2, \cdots$ in a metric space M such that $\sum_1^\infty D(X_i, X_{i+1})$ converges has been called an absolutely convergent sequence by MacNeille [10, p. 192]. Every absolutely convergent sequence is a Cauchy sequence; conversely, every Cauchy sequence contains an absolutely convergent subsequence. Thus there is no loss in generality in assuming that all Cauchy sequences are absolutely convergent. From the relations in (7.7), the properties of exterior measure, and the fact that $\mathfrak{M}$ is a σ-system and a δ-system it follows that M is a complete metric space. The limit of the absolutely convergent sequence $X_1, X_2, \cdots$ is $(X_1 X_2 \cdots) + (X_2 X_3 \cdots) + \cdots$ or $(X_1 + X_2 + \cdots)(X_2 + X_3 + \cdots) \cdots$, these two sets being equal in M. This proof seems to be simpler than those known for a less general result (see Nikodym [12, pp. 139-140], Wazewski [17], and Fréchet [4]). The relations (7.7) enable us to generalize easily other results of Wazewski [17].

Let $\mathfrak{M}_L \subset \mathfrak{M}$ denote the system of Lebesgue measurable sets on $0 \leqq x \leqq 1$, and M_L the corresponding subset of M. The metric space M_L has been the subject of numerous investigations (see Fréchet [4], Nikodym [12], Szpilrajn [15, 16], Wazewski [17]). Since $\mathfrak{M}_L$ is also a σ-system and a δ-system, it follows from (7.7) that M_L is a closed subset of M. It is easily shown that M_L is separable (see Wazewski [17]), but it is known that there exist completely additive measure functions for which the corresponding metric space is not separable (see Nikodym [13]).

In the space M_L every Cauchy sequence $X_1, X_2, \cdots$ converges to a limit X and has an absolutely convergent subsequence $X_{n_1}, X_{n_2}, \cdots$ such that $D(\overline{\lim}_{k\to\infty} X_{n_k}, \underline{\lim}_{k\to\infty} X_{n_k}) = 0$, $D(\overline{\lim}_{n\to\infty} X_{n_k}, X) = 0$. Furthermore, if $X_1, X_2, \cdots$ is a sequence such that $D(\overline{\lim}_{n\to\infty} X_n, \underline{\lim}_{n\to\infty} X_n) = 0$, the sequence is a Cauchy sequence. Since the sequence $\{X_n + [\overline{\lim}_{n\to\infty} X_n - \underline{\lim}_{n\to\infty} X_n]\}$ is metrically the same as the given sequence, there is no loss of generality in assuming that the sets $\overline{\lim}\, X_n$ and $\underline{\lim}\, X_n$ are the same. From (7.7) we have

$$s(X_n, \overline{\lim} X_n) \subseteq s(X_n, X_{n+1}) + s(X_{n+1}, X_{n+2}) + \cdots,$$
$$s(X_n, \underline{\lim} X_n) \subseteq s(X_n, X_{n+1}) + s(X_{n+1}, X_{n+2}) + \cdots.$$

Denote the set on the right by Y_n. Then $Y_1 \supseteq Y_2 \supseteq \cdots$, and the product of these sets is empty. It follows from the definition of distance that $D(X_n, \overline{\lim} X_n) \leqq m[Y_n]$, $D(X_n, \underline{\lim} X_n) \leqq m[Y_n]$. Since $\lim m[Y_n] = 0$, it follows that $\lim X_n = \overline{\lim} X_n = \underline{\lim} X_n$, and hence that $X_1, X_2, \cdots$ is a Cauchy sequence in M_L.

Finally, M_L is convex in the sense of Menger and quasi convex in the sense of Blanc (see Menger [11], Blumenthal [2, p. 40], and Blanc [1]). Let X_1, X_2 be two distinct elements in M_L; we shall show that there is an element in M_L which is between X_1 and X_2. If $D(X_1, X_1 + X_2) \neq 0$, $D(X_1 + X_2, X_2) \neq 0$, it follows from the definition of betweeness, from the properties of Lebesgue measure, and from the relations in (7.8) that $X_1 + X_2$ is in M_L, and that $D(X_1, X_1 + X_2) + D(X_1 + X_2, X_2) = D(X_1, X_2)$, that is, that $X_1 + X_2$ is between X_1 and X_2. Suppose next that $D(X_1, X_1 + X_2) = 0$. Then X_2 is contained in X_1 except possibly for a set of measure zero, and $X_1 - X_2$ is a set of positive measure. In this case there is a measurable set X_3 which contains X_2 and is contained in X_1 and differs from both by sets of positive measure. As before, we can show that X_3 is between X_1 and X_2. The remaining case, in which $D(X_1 + X_2, X_2) = 0$, is similar. It follows from general theorems that any two elements of M_L can be joined by a segment (see Blumenthal [2, p. 41]); the quasi convexity of M_L is a corollary of this fact.

Let $\mathfrak{M}$ be any σ-system and δ-system contained in a space in which there is an exterior measure; then, as above, $\mathfrak{M}$ can be metricised in such a way that it becomes a complete metric space M.

The University of Kansas,
Lawrence, Kansas.

REFERENCES

1. E. Blanc, "Les espaces métriques quasi convexes," *Annales Scientifiques de L'École Normale Supérieure*, 3d series, vol. 55 (1938), pp. 1-82.
2. L. M. Blumenthal, "Distance Geometries: A Study of the Development of Abstract Metrics," *The University of Missouri Studies*, Columbia, Missouri, 1938.
3. M. Fréchet, *Les Espaces Abstraits*, Gauthier-Villars, Paris, 1928.
4. ———, "Sur la distance de deux ensembles," *Comptes Rendus*, vol. 176 (1923), pp. 1123-1124.

5. A. H. Frink, "Distance functions and the metrization problem," *Bulletin of the American Mathematical Society*, vol. 43 (1937), pp. 133-142.
6. H. Hahn, *Reelle Funktionen*, Akademische Verlagsgesellschaft, Leipzig, 1932.
7. F. Hausdorff, *Mengenlehre*, Walter de Gruyter, 2d edition, Berlin and Leipzig, 1927.
8. L. V. Kantorovitch, "Lineare halbgeordnete Räume," *Recueil Mathématique*, new series vol. 2 (1937), pp. 121-165.
9. C. Kuratowski, *Topologie I*, Monografje Matematyczne, vol. III, Warszawa, 1933.
10. H. M. MacNeille, "Extensions of measure," *Proceedings of the National Academy of Sciences*, vol. 24 (1938), pp. 188-193.
11. K. Menger, "Untersuchungen über allgemeine Metrik," *Mathematische Annalen*, vol. 100 (1928), pp. 75-163.
12. O. Nikodym, "Sur une généralization des intégrales de M. J. Radon," *Fundamenta Mathematicae*, vol. 15 (1930), pp. 131-179.
13. ———, "Sur l'existence d'une mesure parfaitement additive et non séparable," *Académie Royale de Belgique. Classe des Sciences. Mémoires*, vol. 17 (1938), fascicule 8.
14. W. Sierpiński, *Introduction to General Topology*, University of Toronto Press, Toronto, 1934.
15. E. Szpilrajn, "The characteristic function of a sequence of sets and some of its applications," *Fundamenta Mathematicae*, vol. 31 (1938), pp. 207-223.
16. ———, "On the space of measurable sets," *Annales de la Société Polonaise de Mathématique*, vol. 18 (1938), pp. 120-121.
17. T. Wazewski, "Sur les ensembles mesurables," *Comptes Rendus*, vol. 176 (1923), pp. 69-70.

ON REPRESENTATIONS OF CERTAIN FINITE GROUPS.*

By EUGENE P. WIGNER.

1. The purpose of this paper is the derivation of a classification of the representations of finite groups with special reference to groups which satisfy the following two conditions:

a. Every element is equivalent to its reciprocal, i. e., all classes are ambivalent.

b. The Kronecker (or "direct") product of any two irreducible representations of the group contains no representation more than once.

Groups of this character will be called S. R. groups (simply reducible). The symmetric permutation groups of the third and fourth degree, the quaternion group, the three dimensional rotation group, the two dimensional unimodular unitary group are S. R. groups. The significance of condition *a* is that every representation is equivalent to the conjugate imaginary representation. One sees this most easily by assuming the representation to be unitary. Then, the traces of reciprocal elements are conjugate complex. They are, on the other hand, equal, since they belong to the same class. Thus all traces are real and conjugate complex representations are equivalent.

The groups of most eigen-value problems occurring in quantum theory are S. R. This is important for the following reason.

Let us assume that we have two eigen-value problems $H_1\psi_1 = \lambda_1\psi_1$ and $H_2\psi_2 = \lambda_2\psi_2$ which allow the same group; ψ_1 and ψ_2 shall be defined in different spaces. One often considers then [1] the "united system" the wave functions Ψ of which are defined in the product space of the spaces of ψ_1 and ψ_2. The "unperturbed" eigen-value equation is $(H_1 + H_2)\Psi = \Lambda\Psi$. The multiplicity of the eigen-value $\Lambda = \lambda_1 + \lambda_2$ is the product of the multiplicities of λ_1 and λ_2. The eigen-value Λ splits up if one introduces a small perturbation term into the last equation. If this perturbation allows the same group as the original two problems and if this group satisfies the above condition *b*, the characteristic functions of the eigen-values into which Λ splits can be determined in "first approximation" by the invariance of the eigen-value problem under the group. The properties of S. R. groups to be derived here

* Received May 1, 1940.

[1] For a more complete discussion cf. e. g. E. Wigner: *Gruppentheorie*, etc., Braunschweig, 1931.

give a basis for a suitable normalization of (and numerous relations between) these eigen-functions which will be dealt with elsewhere.

We shall denote the different irreducible representations of a group by letters j, k, l, etc. The identical representation (in which the matrix (1) corresponds to every element) by 0. The elements of the group will be P, Q, R, S, T, etc. The rows and columns of the representations will be designated by small Greek letters κ, λ, μ, ν, etc. The $\kappa\lambda$ element of the matrix which corresponds in the j-th representation to the element R will be denoted by $\begin{bmatrix} jR \\ \kappa\lambda \end{bmatrix}$ so that one has

$$\sum_{\lambda} \begin{bmatrix} jR \\ \kappa\lambda \end{bmatrix} \begin{bmatrix} jS \\ \lambda\mu \end{bmatrix} = \begin{bmatrix} j\,RS \\ \kappa\ \ \mu \end{bmatrix} \tag{1}$$

The character will be abbreviated to

$$\sum_{\kappa} \begin{bmatrix} jR \\ \kappa\kappa \end{bmatrix} = [j;R]. \tag{1a}$$

The summation over the indices κ, λ etc. referring to the rows or columns of the representations will always run over all values. The unit element of the group will be E, the degree of the representations j is

$$[j;E] = [j]. \tag{1b}$$

A star will denote the conjugate complex. The Kronecker product of the representations j and k coordinates to the group element R the matrix $M_{\kappa\mu;\lambda\nu}$ the rows and columns of which are denoted by double indices $\kappa\mu$ and $\lambda\nu$ respectively. We set

$$M_{\kappa\mu;\lambda\nu} = \begin{bmatrix} jR \\ \kappa\lambda \end{bmatrix} \begin{bmatrix} kR \\ \mu\nu \end{bmatrix}. \tag{2}$$

The character corresponding to the element R is

$$\sum_{\kappa\mu} M_{\kappa\mu;\kappa\mu} = \sum_{\kappa\mu} \begin{bmatrix} jR \\ \kappa\kappa \end{bmatrix} \begin{bmatrix} kR \\ \mu\mu \end{bmatrix} = [j;R][k;R]. \tag{2a}$$

The significance of condition b for the groups under consideration becomes evident if one reduces the Kronecker product of two representations, i. e. brings it into the form in which it appears as the sum of irreducible representations. The matrix by which this transformation can be carried out is—apart from some phase factors—uniquely determined.

It may be useful to write down the well known orthogonality and completeness relations for irreducible representations. These are

$$\sum_{R} \begin{bmatrix} jR \\ \kappa\lambda \end{bmatrix}^* \begin{bmatrix} kR \\ \mu\nu \end{bmatrix} = \frac{h}{[j]} \delta_{jk}\delta_{\kappa\mu}\delta_{\lambda\nu} \tag{3}$$

(3a) $$\sum_R [j;R]^*[k;R] = \sum_C n_c[j;C][k;C] = h\delta_{jk}.$$

The summation is to be extended in this and all similar formulas over all group elements;

(4) $$h = \sum_R 1$$

is the order of the group. The summation over C in the second part of (3a) is to be extended over all different classes, n_c is the number of elements of the class C. The completeness relations yield

(5) $$\sum_{j\kappa\lambda} \frac{[j]}{h} \begin{bmatrix} jR \\ \kappa\lambda \end{bmatrix}^* \begin{bmatrix} jS \\ \kappa\lambda \end{bmatrix} = \delta_{R,S}$$

(5a) $$\sum_j [j;R]^*[j;S] = h\,\Delta_{R,S}/n_R = \sum_Q \delta_{RQ,QS}.$$

The summation over j is to be extended over all different irreducible representations, $\delta_{R,S}$ is 1 for $R = S$, zero otherwise, $\Delta_{R,S}$ is 1 if R and S are in the same class, zero otherwise, n_R is the number of the elements of the class of R. All representations are assumed to be in the unitary form, i. e.

(6) $$\sum_\lambda \begin{bmatrix} jR \\ \kappa\lambda \end{bmatrix}^* \begin{bmatrix} jR \\ \mu\lambda \end{bmatrix} = \delta_{\kappa\mu};\ \sum_\kappa \begin{bmatrix} jR \\ \kappa\lambda \end{bmatrix}^* \begin{bmatrix} jR \\ \kappa\nu \end{bmatrix} = \delta_{\lambda\nu}.$$

The irreducible representations can be classified,[2] in general, into three groups: those which can be transformed into a real form, those which cannot but are equivalent to the conjugate complex representation, and those which are not equivalent to the conjugate complex representation. In analogy to the notation customarily used for the two dimensional unimodular unitary group, we shall call the representations of the first kind *integer* representations. Correspondingly $c_j = 1$ will hold for representations j which can be transformed into a real form, $c_j = -1$ will hold for *half integer* representations j which cannot be transformed into a real form but are equivalent to the conjugate complex representation. Finally $c_j = 0$ if the representation j is not equivalent to the conjugate complex of j. According to G. Frobenius and I. Schur [2]

(7) $$\sum_R [j;R^2] = c_j h.$$

2. The number of square roots of an element R will be denoted by $\zeta(R)$

(8) $$\zeta(R) = \sum_S \delta_{R,S^2}.$$

We have

$$\sum_R \zeta(R)^2 = \sum_{R,S} \zeta(R)\delta_{R,S^2} = \sum_S \zeta(S^2) = \sum_{S,T} \delta_{S^2,T^2}$$

---

[2] G. Frobenius and I. Schur, Berl. Ber. 1906, p. 186.

One can replace S by TR in the last summation and obtain

$$\sum_R \zeta(R)^2 = \sum_{R,T} \delta_{TRTR,T^2} = \sum_{R,T} \delta_{R,TR^{-1}T^{-1}}$$

as $TRTR = T^2$ if and only if $R = TR^{-1}T^{-1}$. For a given R, there will be a T such that $R = TR^{-1}T^{-1}$ only if R and R^{-1} are in the same class, i. e. if R is in an ambivalent class. In this case, the number of T satisfying $R = TR^{-1}T^{-1}$ is equal to h/n_R, since each of the n_R members of the class of R is obtained h/n_R times when T runs over all h elements of the group. Hence

$$\sum_R \zeta(R)^2 = \sum_R{}' h/n_R = h. \text{ (number of ambivalent classes)}. \tag{9}$$

The second summation is to be extended only over the elements of the ambivalent classes. The result thus obtained [3] holds for every finite group:

THEOREM 1. *The sum of the squares of the numbers of square roots of all elements of a finite group is equal to the order of the group, multiplied by the number of ambivalent classes.*

All classes are ambivalent in the S. R. groups. Hence

$$\sum_R \zeta(R)^2 = hn \tag{9a}$$

holds for these, where n is the number of all classes.

The number of times the representation i is contained in the Kronecker product of the representations j and k is given by the equation

$$(i, j, k) = \sum_C [i; C]^* [j; C] [k; C] n_c/h \tag{10}$$

where the summation has to be extended, as in (3a) over all classes. Multiplying (7) by $[j; S]$ and summing over j gives, for (5a)

$$\begin{aligned}\sum_j c_j h[j; S] &= \sum_{R,j} [j; R^2][j; S] \\ &= \sum_R \Delta_{R^2,S}\, h/n_S = h\zeta(S).\end{aligned} \tag{11}$$

The $R^2 = S$ equation is satisfied for $\zeta(S)$ group elements R but R^2 is in the class of S for $n_S\zeta(S)$ group elements.

LEMMA 1. *The Kronecker product of two integer representations or of two half integer representations of a S. R. group contains only integer representations; the Kronecker product of an integer and a half integer represen-*

---

[3] This must have been known to the authors of Reference 2 since it follows immediately from a comparison of the last sentence of § 4 with the sentence in italics on page 201.

tation contains only half integer representations. The unitary matrix which transforms an integer representation into the conjugate complex form is symmetric, that which transforms a half integer representation into the conjugate complex form is skew symmetric.[2] Hence the unitary matrix S which transforms the Kronecker product M of two integer or two half integer representations into the conjugate complex form is symmetric:

$$SM = M^*S; \qquad S = S'. \tag{12}$$

If the unitary U brings M into the reduced form $UMU^{-1} = M_r$

$$S_rM_r = M^*{}_rS_r; \qquad S_r = U^*SU^{-1} \tag{12a}$$

and S_r is again symmetric. Since the corresponding parts of M_r and $M^*{}_r$ are equivalent and since M_r does not contain any irreducible representation more than once, S_r is a step matrix just as M_r is and every submatrix of S_r is symmetric on account of the symmetry of S_r. Hence, every submatrix of M_r, i. e. all irreducible parts of M, can be transformed into the conjugate complex form by a symmetric matrix and are integer representations.

If M is the product of an integer and a half integer representation, S will be skew symmetric and the same will hold for S_r and its submatrices. Consequently, all the irreducible parts of M will be half integer representations.

For a S. R. group $c_ic_jc_k = 1$ if (ijk) is different from zero.

Since the (ijk) are positive integers or zero,

$$(ijk)^2 \geq c_ic_jc_k(ijk). \tag{13}$$

The equality sign can hold only if either $(ijk) = 0$, or $(ijk) = 1$ and $c_ic_jc_k = 1$. Hence

$$\sum_{ijk} (ijk)^2 \geq \sum_{ijk} c_ic_jc_k(ijk) \tag{13a}$$

and the equality sign can hold only if for all i, j, k either $(ijk) = 0$ or $(ijk) = 1$ and $c_ic_jc_k = 1$. This is the case, according to the definition of S. R. groups and Lemma 1, for S. R. groups and conversely, if the equality sign holds in (13a), the group must be a S. R. group.

Because of (11), we have

$$\begin{aligned}\sum_{ijk} c_ic_jc_k(i, j, k) &= \sum_{ijk} c_ic_jc_k[i; C]^*[j; C][k; C]n_c/h \\ &= \sum_{ijk} c_ic_jc_k[i; R]^*[j; R][k; R]/h = \sum_R \zeta(R)^3/h\end{aligned}$$

For the left side of (13a) we have, because of (5a)

$$
\begin{aligned}
&\sum_{ijk} (i, j, k)^2 \\
&\qquad = \sum_{ijk} \sum_{CC'} [i; C]^*[j; C][k; C][i; C'][j; C']^*[k; C']^* n_C n_{C'}/h^2 \\
&\qquad = \sum_{ik} \sum_{C} [i; C]^2[k; C]^2 n_C/h = \sum_{C} h/n_C \\
&\qquad = \sum_{R} h/n_R^2 = \sum_{R} v_R^2/h
\end{aligned}
\tag{14a}
$$

other than R itself

where $v_R = h/n_R$ is the number of elements which commute with R. Hence (13a) is equivalent to

THEOREM 2. *The inequality*

$$
\sum_R \zeta(R)^3 \leq \sum_R v_R^2 \tag{15}
$$

holds for every finite group. The equality sign in (15) *holds for all finite S. R. groups and only these.*

3. The Kronecker product of a representation with itself can be decomposed into a symmetric part

$$
B_{\kappa\mu;\lambda\nu} = \frac{1}{2}\begin{bmatrix} jR \\ \kappa\lambda \end{bmatrix}\begin{bmatrix} jR \\ \mu\nu \end{bmatrix} + \frac{1}{2}\begin{bmatrix} jR \\ \kappa\nu \end{bmatrix}\begin{bmatrix} jR \\ \mu\lambda \end{bmatrix} \tag{16}
$$

and an antisymmetric part

$$
A_{\kappa\mu;\lambda\nu} = \frac{1}{2}\begin{bmatrix} jR \\ \kappa\lambda \end{bmatrix}\begin{bmatrix} jR \\ \mu\nu \end{bmatrix} - \frac{1}{2}\begin{bmatrix} jR \\ \kappa\nu \end{bmatrix}\begin{bmatrix} jR \\ \mu\lambda \end{bmatrix} \tag{16a}
$$

It is easy to see that both B and A form a representation of the group. The irreducible parts of both B and A are integer representations in case of $S.$ $R.$ groups. The irreducible parts of the B for integer j and the irreducible parts of the A for half integer j will be called even representations. Conversely, the irreducible parts of the B for integer j will be called odd representations. This notation is taken again from the theory of representations of the two dimensional unitary group.

THEOREM 3. *In S. R. groups no representation can be both even and odd.*

The trace of the symmetric part of the square of the representation j is

$$
X_{js}(R) = \tfrac{1}{2}[j; R]^2 + \tfrac{1}{2}[j; R^2] \tag{17}
$$

and the trace of the antisymmetric part is

$$
X_{ja}(R) = \tfrac{1}{2}[j; R]^2 - \tfrac{1}{2}[j; R^2] \tag{17a}
$$

The condition that two representations have no common part is that the sum of the products of their characters vanish. Theorem 3 is equivalent therefore with the validity of

$$(18) \qquad \tfrac{1}{4} \sum_R ([j;R]^2 + c_j[j;R^2])([k;R]^2 - c_k[k;R^2]) = 0$$

for all j and k. Since the left side of (18) by its nature cannot be negative (it is $\Sigma n_i m_i$ where n_i and m_i are the numbers of times the representation i is contained in the first and second representation of (18)), the validity of (18) for all j and k is equivalent with the vanishing of

$$\begin{aligned}(18a) \qquad & \sum_R \sum_{jk} ([j;R]^2 + c_j[j;R^2])([k;R]^2 - c_k[k:R^2]) \\ & = \sum_R (h/n_R + \zeta(R^2))(h/n_R - \zeta(R^2)) \\ & = \sum_R v_R{}^2 - \sum_R \zeta(R^2)^2\end{aligned}$$

where (5a) and (11) have been utilized. Now evidently

$$\sum_S \zeta(S)^3 = \sum_{S,R} \zeta(S)^2 \delta_{S,R^2} = \sum_R \zeta(R^2)^2$$

so that (18a) vanishes on account of Theorem 2. Hence Theorem 3 is valid.

Of course, the Kronecker product of an even and an odd representation e. g., contains, in general, both even and odd representations. It has not been shown, either, that every integer representation is either even, or odd. In fact, one can easily find a group which has an integer representation which does not occur in the square of any representation. A group of this character is formed by the elements $1, -1, x, -x, y, -y, z, -z$, with the multiplication rules $x^2 = y^2 = 1$, $z^2 = -1$, $xy = -yx = z$, $xz = -zx = y$, $zy = -yz = x$.

PRINCETON UNIVERSITY.

SUBSPACES OF $l_{p,n}$ SPACES.*

By F. Bohnenblust.

1. Introduction. For any Banach space B, the following numbers a_n can be introduced:

$$a_n = \inf \| P \|,$$

where P is any linear projection of B on any subspace of given dimension n. For any Banach space $a_1 = 1$ and $a_n \geqq 1$. When B is the Hilbert space, all $a_n = 1$. If there exists a base in B, then the a_n are bounded. For by a theorem of Schauder,[1] if $x_1, \cdots, x_n, \cdots$ is a base of B,

$$x = \sum_1^\infty \xi_\nu x_\nu,$$

the new norm $||| x ||| = \sup \| \sum_1^n \xi_\nu x_\nu \|$ is isomorphic with the original one: $||| x ||| \leqq C \| x \|$ and the projection P_n, $P_n x = \sum_1^n \xi_\nu x_\nu$, has a norm $\leqq C$. Thus $a_n \leqq C$.

If separable Banach spaces B exist for which the a_n are not bounded, the open question, whether or not every separable Banach space admits a base, will be answered negatively. In the present paper we take a first step in this direction, by constructing simple finite dimensional spaces (of dimension l, say) for which every $a_m > 1$, when $1 < m < l$.

Theorem. *Let $(\xi_1, \cdots, \xi_l)$, ξ_λ real, be a point of an l-dimensional linear space, and let*

$$\| x \| = [\sum_{\nu=1}^n | f_\nu(\xi) |^p]^{1/p},$$

where: (1) *p is a finite real number, $p \neq$ integer;*

(2) $n > 2\,(2l - 3)$;

(3) *the $f_\nu(\xi)$ are linear forms, $f_\nu(\xi) = \Sigma \phi_{\nu\lambda} \xi_\lambda$,*

In general this space will be such that $a_m > 1$, for $1 < m < l$.

('In general' means that the $\phi_{\nu\lambda}$ must satisfy certain relations, which are described in the text). Such a space can be considered more simply as an l-dimensional subspace of $l_{p,n}$, where $l_{p,n}$ is the classical space of elements

* Received May 3, 1940.

[1] E. g. Banach, *Théorie des opérations linéaires*, p. 111.

$(\xi_1, \cdots, \xi_n)$ with the norm $\| x \| = [\Sigma \mid \xi_\nu \mid^p]^{1/p}$. It is in this form that we shall verify our result.

2. Plücker Grassmann coordinates. Let S_m be an m-dimensional linear subspace of the n-dimensional affine space R_n of elements $x = \{\xi(\nu)\}$, $\nu = 1, 2, \cdots, n$. If the subspace S_m is determined by the elements

$$x(\mu) = \{\xi(\mu, \nu)\}, \qquad (\mu = 1, 2, \cdots, m;\ \nu = 1, 2, \cdots, n);$$

the values $p(\nu_1, \nu_2, \cdots, \nu_m)$ of the determinants

$$p(\nu_1, \nu_2 \cdots, \nu_m) = \begin{vmatrix} \xi(1, \nu_1) \cdots \xi(1, \nu_m) \\ \cdots \\ \xi(m, \nu_1) \cdots \xi(m, \nu_m) \end{vmatrix}$$

are the Plücker Grassmann coordinates of the space S_m. Not all of them vanish, and considered as homogenous coordinates, they are determined by S_m independently of the particular choice of the elements $x(\mu)$. Conversely they determine the space S_m uniquely. The Plücker Grassmann coordinates satisfy furthermore the relations [2]

(P) $p(\nu_1, \cdots, \nu_m)$ changes sign when the indices are permuted by an odd permutation,

(R) $$p(\nu_1, \nu_2, N)\, p(\nu_3, \nu_4, N) + p(\nu_1, \nu_4, N)\, p(\nu_2, \nu_3, N) + p(\nu_3, \nu_1, N)\, p(\nu_2, \nu_4, N) = 0$$

for any four indices $\nu_1, \nu_2, \nu_3, \nu_4$ and any set N of $(m - 2)$ indices.

Essentially, these are all the relations the Plücker Grassmann coordinates satisfy. In particular, if $p(\nu_1, \nu_2, \cdots, \nu_m) = 1$, the Plücker Grassmann coordinates

(1.1) $$p(\nu_1, \nu_2, \cdots, \nu_{r-1}, \nu, \nu_{r+1}, \cdots, \nu_m),$$

where ν is any index different from $\nu_1, \nu_2, \cdots, \nu_m$, are independent and determine S_m uniquely, for example, by the elements $x(\mu) = \{\xi(\mu, \nu)\}$

(1.2) $$\begin{aligned} &\xi(\mu, \nu) = \delta_{\mu\mu'}, \text{ if } \nu = \nu_{\mu'}, \\ &\xi(\mu, \nu) = p(\nu_1, \cdots, \nu_{\mu-1}, \nu, \nu_{\mu+1}, \cdots, \nu_m), \text{ otherwise.} \end{aligned}$$

All the subspaces S_m in R_n form thus a locally euclidean, $m(n - m)$-dimensional manifold $\mathfrak{M}$, or more precisely $\mathfrak{M}(n, m)$. In a neighborhood of any point S_m of $\mathfrak{M}$, a set (1.1) of Plücker Grassmann coordinates acts as a set of euclidean coordinates, and a finite number of such neighborhoods cover the entire manifold $\mathfrak{M}$. $\mathfrak{M}$ is an algebraic manifold in the $(n^m - 1)$-dimensional

[2] E. g. Sommerville, *An Introduction to the Geometry of n dimensions.*

projective space $P(n^m - 1)$ of coordinates $p(\nu_1, \cdots, \nu_m)$. (The number of dimensions could easily be reduced, but this will be of no importance to us). A subset of $\mathfrak{M}$ will be called an algebraic manifold in $\mathfrak{M}$, if it is an algebraic manifold in the projective space $P(n^m - 1)$. We shall need the following result, which follows immediately from the relations between the Plücker Grassmann coordinates of a space and those of one of its subspaces:

THEOREM 1.1. *If A is a subset of an algebraic manifold of $\mathfrak{M}(n, m)$, whose dimension is $\leqq a$, then all the points of $\mathfrak{M}(n, l)$, $(l \geqq m)$, which represent subspaces S_l containing at least one subspace S_m belonging to A, form a subset B which lies in an algebraic manifold in $\mathfrak{M}(n, l)$, whose dimension is $\leqq a + (n - l) \cdot (l - m)$.*

3. The sets $G_0, G_1, \cdots, G_k$. Let S_m be an m-dimensional subspace of R_n and let $p(\nu_1, \cdots, \nu_m)$ be its Plücker Grassmann coordinates.

DEFINITION 2.1. *An index ν_0 will be said to belong to G_0, if and only if, $p(\nu_0, N) = 0$ for every set N of $(m - 1)$ indices.*

THEOREM 2.1. *The index ν_0 belongs to G_0, if and only if, the ν_0-th coordinate of every element of S_m vanishes.*

The condition is obviously sufficient. To verify the necessity we observe that at least one Plücker Grassmann coordinate of S_m is different from zero, say $p(1, 2, \cdots, m) = 1$. If ν_0 belongs to G_0, it must be greater than m and the ν_0-th coordinate of the vectors (1.1) vanish. The same holds true for any linear combination of these elements.

DEFINITION 2.2. *Two indices ν_1 and ν_2 of G'_0 (= the complementary set of G_0) will be said to be equivalent, $\nu_1 \sim \nu_2$, if and only if, $p(\nu_1, \nu_2, N) = 0$ for every set N of $(m - 2)$ indices.*

This notion of equivalence is evidently symmetric and reflexive. It is also transitive: assume $\nu_1 \sim \nu$ and $\nu_2 \sim \nu$, but ν_1 and ν_2 not equivalent. There exists then a set N of $(m - 2)$ indices, such that $p(\nu_1, \nu_2, N) = 1$. The index ν cannot belong to N and it follows from (1.1) that ν should belong to G_0, which contradicts the implicit assumption that ν belongs to G'_0.

DEFINITION 2.3. *The sets $G_1, \cdots, G_k$ are the sets into which the notion of equivalence of the preceding definition divides the indices of the complementary set of G_0. The sets $G_0, G_1, \cdots, G_k$ will be referred to as the "type" of the subspace S_m.*

Since at least one Plücker Grassmann coordinate does not vanish, we see

that $m \leqq k$. Furthermore, if $n_0, n_1, \cdots, n_k$ denote the number of indices which are contained in the sets $G_0, G_1, \cdots, G_k$, and if $k_1, k_2, \cdots$ denote the numbers of sets among $G_1, \cdots, G_k$ (i. e. exclusive of G_0) which contain respectively one, two, $\cdots$ indices, the following relations will hold:

$$\begin{aligned} &n_0 \geqq 0, \qquad n_\kappa \geqq 1, \qquad \kappa = 1, 2, \cdots, k; \\ (2.1) \qquad &n_0 + n_1 + \cdots + n_k = n_0 + k_1 + 2k_2 + 3k_3 + \cdots = n; \\ &k_1 + k_2 + \cdots = k. \end{aligned}$$

THEOREM 2.2. *A necessary and sufficient condition that two indices ν_1 and ν_2 of G'_0 be equivalent is that the Plücker coordinate $p(\nu_1, \nu_2)$ of every two-dimensional subspace of S_m be equal to zero.*

Proof. Let ν_1 and ν_2 belong to different sets G. There exist $\nu_3, \nu_4, \cdots, \nu_m$ such that $p(\nu_1, \nu_2, \cdots, \nu_m) = 1$, and thus vectors $x(\mu) = \{\xi(\mu, \nu)\}$ in S_m for which

$$(2.2) \qquad \xi(\mu, \nu_{\mu'}) = \delta_{\mu\mu'}.$$

The vectors $x(1)$ and $x(2)$ determine then a two-dimensional subspace whose $p(\nu_1, \nu_2) = 1$. This shows the sufficiency. Assume now the existence of a two-dimensional subspace with $p(\nu_1, \nu_2) = 1$. There exist in this subspace two vectors $x(1)$ and $x(2)$ satisfying equations (2.2) for $\mu, \mu' = 1, 2$. They are linearly independent and can be completed by $x(3), x(4), \cdots, x(m)$ to form a base for S_m. In addition, we may assume equations (2.2) to be satisfied for $\mu > 2$, $\mu' = 1, 2$ and it is then readily seen that at least one $p(\nu_1, \nu_2, N)$ of S_m does not vanish, i. e. that the indices ν_1 and ν_2 are not equivalent. Theorems 2.1 and 2.2 imply then the

THEOREM 2.3. *The type of any subspace of a subspace S_m is an aggregation of the type of S_m; i. e. every set of the type of S_m is contained entirely in one of the sets of the type of the subspace of S_m. In particular the set G_0 of S_m is contained in the set G_0 of the subspace.*

We complete this theorem by proving next

THEOREM 2.4. *In every S_m, there exists a two-dimensional (and thus of any dimension $< m$, but > 1) subspace of the same type as S_m.*

Proof. Choose from each set of the type of S_m one index. Let us denote these by $\nu_1^0, \cdots, \nu_k^0$. Let $x(\mu) = \{\xi(\mu, \nu)\}$ be a base of S_m and let $\rho_1, \cdots, \rho_m$; $\sigma_1, \cdots, \sigma_m$ be independent variables. The Plücker coordinates of the subspace determined by the two vectors $\Sigma\rho_\mu \cdot x(\mu)$ and $\Sigma\sigma_\mu \cdot x(\mu)$ are bi-linear in these variables and by theorem 2.2 none of the $p(\nu_1^0, \nu_2^0), \cdots, p(\nu_1^0, \nu_k^0)$,

$\cdots, p(\nu^0_{k-1}, \nu_k^0)$ are identically zero. There exist ρ_μ, σ_μ such that none of them vanishes, and the type of this subspace coincides with the type of S_m.

THEOREM 2.5. *If S_m is of type $G_0, \cdots, G_k$ and if $\nu_1^0, \cdots, \nu_k^0$ are indices belonging to $G_1, \cdots, G_k$ respectively, there exist real numbers $\alpha(\nu)$ defined for ν in G'_0, and in the k-dimensional affine space R_k an m-dimensional subspace T_m with Plücker Grassmann coordinates $q(\kappa_1, \cdots, \kappa_m)$ such that*

(1) $$\alpha(\nu) \neq 0, \text{ for } \nu \text{ in } G'_0,$$

(2) $$\alpha(\nu_\kappa^0) = 1,$$

(3) *for any indices ν_μ in G'_0*

$p(\nu_1, \cdots, \nu_m) = \alpha(\nu_1) \cdots \alpha(\nu_m)\, q(\kappa_1, \cdots, \kappa_m)$, *where κ_μ is the index of the set G which contains ν_μ,*

(4) *The type of T_m is: G_0 is void, every other G has exactly one index.*

Proof. Let $b = \{\beta(\nu)\}$ be a vector of S_m for which $\beta(\nu)$ is different from zero when ν lies in G'_0. (The existence of such a vector b follows immediately from the definition of G_0). Define, for ν in G'_0, the $\alpha(\nu)$ by the equations $\alpha(\nu) = \beta(\nu)/\beta(\nu_\kappa^0)$, where ν_κ^0 is the index corresponding to the set which contains ν. Let, finally, $\nu_1 \sim \nu'_1$ and let $\nu_2, \cdots, \nu_m$ be any set of $(m-1)$ indices of G'_0. Since the vector b lies in S_m, and since $p(\nu_1, \nu'_1, N) = 0$, we have the relation

$$\beta(\nu_1) \cdot p(\nu'_1, \nu_2, \cdots, \nu_m) = \beta(\nu'_1) \cdot p(\nu_1, \nu_2, \cdots, \nu_m)$$

i. e.

$$\frac{p(\nu'_1, \nu_2, \cdots, \nu_m)}{\alpha(\nu'_1) \cdot \alpha(\nu_2) \cdots \alpha(\nu_m)} = \frac{p(\nu_1, \nu_2, \cdots, \nu_m)}{\alpha(\nu_1)\alpha(\nu_2) \cdots \alpha(\nu_m)}.$$

By an induction proof, we see that

$$\frac{p(\nu'_1, \nu'_2, \cdots, \nu'_m)}{\alpha(\nu'_1) \cdot \alpha(\nu'_2) \cdots \alpha(\nu'_m)} = \frac{p(\nu_1, \nu_2, \cdots, \nu_m)}{\alpha(\nu_1) \cdot \alpha(\nu_2) \cdots \alpha(\nu_m)},$$

if $\nu_\mu \sim \nu'_\mu$. In other words we can put

$$p(\nu_1, \cdots, \nu_m) = \alpha(\nu_1) \cdots \alpha(\nu_m) \cdot q(\kappa_1, \cdots \kappa_m).$$

If the ν_μ differ from each other and happen to be chosen among the indices $\nu_1^0, \cdots \nu_\kappa^0$, all the corresponding α are equal to one and $p(\cdots) = q(\cdots)$. Since, furthermore, at least one of these p does not vanish, the numbers q are the Plücker Grassmann coordinates of a subspace T_m in R_k. The statements (1), (2), (3) of the theorem have thus been verified. To verify the last one we remark first, that if $q(\kappa, K)$ is always equal to zero for any K, the co-

ordinates $p(\nu_\kappa{}^0, N)$ will all be zero, which is impossible since $\nu_\kappa{}^0$ does not belong to G_0. Thus the set G_0 of T_m must be void. Secondly, if $q(\kappa_1, \kappa_2, K) = 0$, $(\kappa_1 \neq \kappa_2)$, for any K, we have similarly $p(\nu_{\kappa_1}{}^0, \nu_{\kappa_2}{}^0, N) = 0$ which implies $\nu_{\kappa_1}{}^0 \sim \nu_{\kappa_2}{}^0$, whereas they belong to different sets G. Thus the other sets of the type of T_m can contain only one element.

We remark in passing that the converse of the theorem is also true, and also that a vector $x = \{\xi(\nu)\}$ belongs to S_m, if and only if, it is of the form

$$\xi(\nu) = 0 \text{ if } \nu \in G_0,$$
$$\xi(\nu) = \alpha(\nu) \cdot \eta(\kappa) \text{ if } \nu \text{ in } G'_0,$$

where κ corresponds to ν and $y = \{\eta(\kappa)\}$ is a vector of T_m.

The coordinates of any subspace S_m of a given type $G_0, \cdots, G_k$ are thus expressed as polynomials in terms of $k_2 + 2k_3 + \cdots = n - n_0 - k$ variables α and certain q which lie in the $m(k-m)$-dimensional algebraic manifold $\mathfrak{M}(k, m)$. These subspaces S_m will lie, therefore, in an algebraic manifold $A(m, G_0, \cdots G_k)$ in $\mathfrak{M}(n, m)$ of dimension $\leqq n - n_0 - k + m(k-m)$. The union of algebraic manifolds is an algebraic manifold, and thus for a given integer k, the S_m which are of a type $G_0, \cdots, G_k$ will lie in an algebraic manifold $A(m, k)$ of dimension $\leqq n - k + m(k-m)$.

Let l be an integer $\geqq m$. Define $B(m, l)$ as the union of those $A(m, k)$ for which $(m-1)k < m(n-l) + m^2 - n$, with the understanding that B is void if there are no k satisfying this inequality. By theorem 1. 1, the points S_l of $\mathfrak{M}(n, l)$ which are subspaces S_l containing at least one subspace S_m which belongs to $B(m, l)$, will lie in an algebraic manifold $C(m, l)$ of dimension $< m(n-l) + (n-l)(l-m)$. The dimension of the union $C(l) = C(2, l) + \cdots + C(l, l)$ is $< l(n-l)$, i. e. less than the dimension of $\mathfrak{M}(n, l)$.

DEFINITION 2. 4. *A subspace S_l in R_n will be said to be in general position, if all Plücker Grassmann coordinates of S_l are different from zero and if S_l does not lie in $C(l)$.*

The existence of S_l in general position follows from our consideration of the dimension of $C(l)$ and the fact that the S_l, for which one Plücker Grassmann coordinate vanishes, form also a manifold of dimension $< l(n-l)$. The construction of $C(l)$ allows us to state the following theorem.

THEOREM 2. 6. *Let S_l be a subspace of R_n in general position. The number k of the type of any subspace S_m of S_l $(2 \leqq m \leqq l-1)$ satisfies the inequality*

$$(m-1)\, k \geqq m(n-l) + m^2 - n.$$

Since $2k \leqq k_1 + (k_1 + 2k_2 + 3k_3 + \cdots) \leqq k_1 + n$ we obtain from the last theorem that

$$k_1 \geqq n - \frac{2m}{m-1}(l-m).$$

If we assume furthermore that $n > 2(2l-3)$, the last inequality takes on the form, substituting for $m/(m-1)$ its maximal value 2,

$$k_1 > 4m - 6 = m + 3(m-2) \geqq m.$$

In other words we have obtained the following result.

THEOREM 2.7. *If S_l is in general position in an affine space of dimension $> 2(2l-3)$, the number k_1 of the sets of the type of any subspace of S_l (whose dimension is at least 2), which contain only one index, is greater than the dimension of the subspace of S_l.*

THEOREM 2.8. *Let p be any real number > 1 and not an integer. Let S_m in R_n be of the type $k = n$, i. e. G_0 is void, and all other G contain exactly one index. If*

$$\sum \gamma(\nu) \mid \xi(\nu) \mid^{p-1} \operatorname{sign} \xi(\nu)$$

vanishes for every vector $x = \{\xi(\nu)\}$ of S_m, then every $\gamma(\nu)$ must be equal to zero.

Proof. By theorem 2.4 there exists a two-dimensional subspace S_2 of S_m of the same type as S_m. Let $x^0 = \{\xi^0(\nu)\}$ and $x^1 = \{\xi^1(\nu)\}$ be two linearly independent vectors of S_2, where we assume in addition that $\xi^0(\nu) \neq 0$ for every ν. (Since G_0 is void, this is no restriction). For every real λ

$$\sum \gamma(\nu) \mid \xi^0(\nu) + \lambda\xi^1(\nu) \mid^{p-1} \operatorname{sign} (\xi^0(\nu) + \lambda\xi^1(\nu))$$

vanishes by assumption. For $\mid \lambda \mid < \operatorname{Min} [\mid \xi^0(\nu)/\xi^1(\nu) \mid]$, each term is analytic in λ and by evaluating the derivatives at $\lambda = 0$, we obtain for any non-negative integer h the relations

$$\sum \gamma(\nu) \cdot \mid \xi^0(\nu) \mid^{p-1} (\xi^1(\nu)/\xi^0(\nu))^h \operatorname{sign} \xi^0(\nu) = 0.$$

Let h assume the values $0, 1, 2, \cdots, n$. The determinant of the $(n+1)$ linear relations thus obtained is equal to

$$\prod_\nu \frac{\mid \xi^0(\nu) \mid^p}{\xi^0(\nu)^n} \cdot p(1,2) \cdot p(1,3) \cdot \cdots \cdot p(n-1, n)$$

and thus different from zero, since S_2 is of the type $k = n$. These relations admit therefore only the trivial solution.

THEOREM 2.7. *A necessary and sufficient condition that*

$$\sum \gamma(\nu) \mid \xi(\nu) \mid^{p-1} \operatorname{sign} \xi(\nu)$$

vanishes for every element $x = \{\xi(\nu)\}$ *of* S_m *is that*

$$\sum_{\nu \text{ in } G_\kappa} \gamma(\nu) \mid \alpha(\nu) \mid^{p-1} \operatorname{sign} \alpha(\nu) = 0 \qquad \text{for } \kappa = 1, \cdots, k.$$

Proof. Substitute for $\xi(\nu)$ their expressions in terms of the α and the coordinates of a vector of the space T_m of theorem 2.5 and apply theorem 2.6 to the space T_m.

3. Projections of norm one. Let S be a Banach space, such that to every element x of S different from 0, there exists only one functional f_x of norm 1 for which $f_x(x) = \|x\|$. Let S' and S'' be closed subspaces of S, S' contained in S''. Let P be a projection of norm one of S'' onto S'. If $x \neq 0$ is an element in S' and y any element in S'', the linear functional g defined in S'' by $g(y) = fx(Py)$ has a norm equal to one. It can be extended to S without increasing its norm. Since $g(x) = \|x\|$, we have by assumption $g(y) = f_x(Py)$, i. e. the relations

$$f_x(Py) = f_x(y) \text{ for any } x \neq 0, x \text{ in } S' \text{ and any } y \text{ in } S''. \tag{3.1}$$

Conversely, if this condition is satisfied for a projection P of S'' onto S', the norm of P must be one. For, let y be any element of S'' such that Py is not the origin. We have then

$$\|Py\| = f_{Py}(Py) = f_{Py}(y) \leqq \|y\|.$$

It is well known that the l_p spaces, and in particular the $l_{p,n}$ spaces, satisfy the condition imposed on S, provided $1 < p < \infty$. The functional f_x is given by

$$f_x(y) = \{\sum \mid \xi(\nu) \mid^{p-1} \operatorname{sign} \xi(\nu) \cdot \eta(\nu)\} / \|x\|^{p-1}, \tag{3.2}$$

where $x = \{\xi(\nu)\}$ and $y = \{\eta(\nu)\}$, $\nu = 1, \cdots, n$. If $S' = S_m$ is of type $G_0, G_1 \cdots, G_k$ and $S'' = S_l$ equations (3.1) and (3.2) show that for every y in the $(l - m)$-dimensional subspace where $P = 0$

$$\sum \mid \xi(\nu) \mid^{p-1} \operatorname{sign} \xi(\nu) \cdot \eta(\nu) = 0 \tag{3.3}$$

i. e.[3]

$$\sum_{\nu \text{ in } G_\kappa} \mid \alpha(\nu) \mid^{p-1} \operatorname{sign} \alpha(\nu) \cdot \eta(\nu) = 0 \qquad \text{for } \kappa = 1, \cdots, k. \tag{3.4}$$

[3] We assume p not an integer.

Conversely, if S_m is given, if U is the subspace determined by equations (3.4), and if V is the subspace spanned by S_m and U, there exists a projection of norm one of V onto S_m. The subspaces S_m and U have only the origin in common. For, if x is in S_m we have $\xi(\nu) = 0$ for ν in G_0 and $\xi(\nu) = \alpha(\nu) \cdot \eta(\kappa)$ otherwise. Substituting these values in (3.4) we see that every $\eta(\kappa)$ must vanish, and consequently that $x = 0$. Let P be the projection of V onto S_m, which projects U into the origin. For this projection the equations (3.1) are satisfied and the norm of P must thus be zero. Hence we obtain the following result.

THEOREM 3.1. *A necessary and sufficient condition that there exists a projection of norm one of S_l onto S_m is that S_l be contained in the subspace spanned by S_m and the subspace U determined by the equations* (3.4).

In the particular case where S_l is the entire space R_n it is necessary and sufficient to verify that the dimension of U is equal to $n - m$.

THEOREM 3.2. *A necessary and sufficient condition that there exist a projection of norm one of $l_{p,n}$ on S_m is that $k = m$, i. e. that the type of S_m contains exactly m sets besides possibly a set G_0.*

We return to the general case, where S_m need not be equal to R_n. The subspace U lies in the $(n - k_1)$-dimensional subspace W defined by $\xi(\nu) = 0$ for ν in a set G_κ $(\kappa = 1, 2, \cdots, k)$ which contains only one index. If S_m is in general position in R_n, its Plücker Grassmann coordinates are all different from zero and the intersection of S_l and W is exactly $(l - k_1)$ dimensional. If there exists a projection of norm 1 of S_l onto S_m, the dimension of U and a fortiori of W must be at least $l - m$, i. e. $k_1 \leqq m$. Comparing this statement with theorem 2.7 we obtain the theorem.

THEOREM 3.3. *In a subspace S_l in general position in $l_{p,n}$ (p finite, not equal to an integer), where $n > 2(2l - 3)$, only the identity and projections on one dimensional subspaces can have the norm one.*

PRINCETON UNIVERSITY AND
CALIFORNIA INSTITUTE OF TECHNOLOGY.

ON GENERALIZED RINGS.*

By David C. Murdoch and Oystein Ore.

The present paper contains an axiomatic investigation of the conditions underlying the main theorems in the theory of rings. In many results it is related to the axiomatic study of the properties of groups given by *Hausmann* and *Ore*.[1] One considers, to begin with, a system with two operations, conveniently called addition and multiplication. The ideals in this system are defined by the ordinary ideal properties and they must also be normal subsystems with respect to the additive operation, normality being suitably defined.

We consider first the conditions for the ordinary decomposition theorems of ideal theory to hold, i. e., the conditions for the ideals to form a Dedekind structure. It is found that in order to obtain this property the system has to satisfy conditions which are almost equivalent to the group axioms with respect to addition. Hence we proceed to the study of *generalized rings,* which are systems forming a group, usually non-commutative, with respect to addition. For the ideals in a general ring the Dedekind condition is found to be satisfied, provided only that a weak distributive law holds for multiplication.

Next the conditions for the various properties of ideal multiplication are derived. Similarly the axiomatic conditions for the residuals or ideal quotients are obtained. One finds that the various properties of the residuals fall into classes, each dependent upon some particular property of the ring multiplication.

The present investigation seems useful as it has already been stated, because it makes clear the essential conditions for each part of ideal theory. This makes it possible as soon as a system with two operations is given, to determine which conditions are satisfied. This appears to be a more satisfactory approach than the usual one, in which each ring generalization is studied separately with respect to its ideal properties. Because of the generality of the theory, it becomes possible to include well-known systems like the Lie algebras. A still more important case is perhaps the theory of ordinary groups. Any group may be made into a generalized ring by making the group multiplication the addition in the ring, and introducing the commutator as the product of two elements. One finds then that the ordinary theory of normal subgroups becomes identical with the ideal theory in the ring. One also obtains com-

* Received February 12, 1940.

[1] B. A. Hausmann and Oystein Ore: "Theory of quasi-groups," *American Journal of Mathematics*, vol. 59 (1937), pp. 983-1003.

mutator products and commutator residuals in the group, and various group properties follow directly from the general properties of residuals.

CHAPTER I. Conditions for decomposition theorems.

1. Ordinary rings. A *ring* $\mathfrak{R}$ is usually defined as an algebraic system which is closed under two operations, *addition* and *multiplication*. About these two operations one usually makes the following assumptions.

Ring axioms.

1. The elements of $\mathfrak{R}$ form an Abelian group with respect to addition.

2. Multiplication satisfies the associative and distributive laws. Furthermore certain other properties like existence of units, cancellation laws or the commutative law may be postulated.

One of the main problems in such a ring is the study of the properties of its ideals. An ideal is then defined as a subsystem of the ring which satisfies the conditions:

Ideal conditions.

1. If the ideal $\mathfrak{a}$ contains a_1 and a_2 then it contains $a_1 \pm a_2$.

2. If r is any element of $\mathfrak{R}$ and a any element in $\mathfrak{a}$, then if $\mathfrak{a}$ is a left (right) ideal it contains ra (ar). If it contains both ra and ar the ideal is said to be *two-sided.*

In the following we shall write $\mathfrak{a} \cap \mathfrak{b}$ for the set of common elements of the two ideals or even arbitrary sets $\mathfrak{a}$ and $\mathfrak{b}$. The union $\mathfrak{a} \cup \mathfrak{b}$ of two (left, right, two-sided) ideals shall denote the smallest ideal of the same type containing $\mathfrak{a}$ and $\mathfrak{b}$.

Before we proceed let us make a few remarks as to the systematic meaning of the preceding ideal conditions. The first condition shows that the ideal is a modulus, i. e., a subgroup of the additive group. The set of all such subgroups form a *structure* Σ_M, and this structure satisfies the *Dedekind axiom* since the group is Abelian.

The second ideal condition implies first that any ideal must be a subring. Now all subrings of $\mathfrak{R}$ form a structure Σ_R. The elements of Σ_R belong to Σ_M but Σ_R is not a proper substructure of Σ_M in the sense that the definition of the structure operations coincide, since the union $\mathfrak{R}_1 \cup \mathfrak{R}_2$ of two subrings is the smallest ring containing $\mathfrak{R}_1$ and $\mathfrak{R}_2$ while their union $\mathfrak{R}_1 + \mathfrak{R}_2$ in Σ_M is the smallest modulus containing both. Also Σ_R will usually not satisfy the Dedekind axiom. One sees however that the requirement of the second ideal

condition is sufficiently strong to make the set of ideals a Dedekind structure Σ_I and this structure is a proper substructure of Σ_M with $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{a} + \mathfrak{b}$ for any two ideals.

2. Generalized rings. We shall now turn to the generalizations of the ring concept. A very broad extension of the concept is the following.

Algebraic system with two operations:

In the system $\mathfrak{R}$ there shall exist two operatons, addition and multiplication such that $a + b$ and $a \cdot b$ are uniquely defined elements of the system.

In this definition no conditions on the operations are assumed. However, in this general form there is but little which can be proved for the system. We shall therefore, at least for the moment, limit our considerations to a more specialized system, which covers most of the important applications:

Generalized ring.

A generalized ring is an algebraic system with two operations, addition and multiplication, forming a group with respect to addition.

Let us observe that this definition assumes nothing about multiplication except that it exists. A subring may be defined as a subgroup of the additive group which is closed with respect to multiplication. The cross-cut of two subrings is the set of common elements. It cannot be void since it must contain the additive zero element. The union of two rings consists of all combinations of sums and products of the elements in the two rings. The subrings are seen to form a structure as before.

The preceding discussion shows that it is natural to adopt the following definition of an ideal in a generalized ring:

Generalized ideal conditions.

1. An ideal $\mathfrak{a}$ is a normal subgroup of the additive group.
2. For a left (right) ideal $r \cdot \mathfrak{a} \subset \mathfrak{a}$ $(\mathfrak{a} \cdot r \subset \mathfrak{a})$ for every element r in $\mathfrak{R}$.

Since the cross-cut of two normal subgroups is again a normal subgroup it is obvious that the cross-cut $\mathfrak{a} \cap \mathfrak{b}$ of two ideals is again an ideal. Next we turn to the union $\mathfrak{a} \cup \mathfrak{b}$ of the same ideals. This union must contain all sums of elements in the two ideals and since $\mathfrak{a}$ and $\mathfrak{b}$ are normal subgroups of the additive group all such sums are of the form $a + b$. Now the second condition states that $c(a + b)$ shall also be in the ideal for any c in $\mathfrak{R}$. To create an analogy to the ordinary ideal theory and in particular to prove the Dedekind relation one is compelled to assume a distributive law of the form:

Distributive Law I. *For any elements a, b, c, $c(a + b) = a_1 + b_1$ where a_1 and b_1 belong to the left ideals generated by a and b respectively.*

To fix the ideas let us consider the properties of left ideals, hence we shall use the Distributive Law I for such ideals.

There are various special cases of this distributive law which may be mentioned separately. One case is $c(a+b) = c_1 \cdot a_1 + c_2 \cdot b_1$ where c_1 and c_2 are arbitrary while a_1 and b_1 belong to the ring, or still more specially, to the additive group defined by a and b respectively. Another special case is $c(a+b) = c_1 \cdot a_1^r + c_2 \cdot b_1^s$ where c_1, c_2, a_1 and b_1 may have the same meaning as before while the exponents r and s indicate the transformation by these elements in the additive group $a^r = r + a - r$.

We can now show:

THEOREM 1. *In a generalized ring the left (right) ideals form a Dedekind structure when the Distributive Law I holds.*

Proof. It remains only to deduce the Dedekind relation

$$\mathfrak{c} \cap (\mathfrak{a} \cup \mathfrak{b}) = \mathfrak{a} \cup (\mathfrak{c} \cap \mathfrak{b}), \mathfrak{c} \supset \mathfrak{a}$$

for the ideals. It is obvious that the right-hand side is always contained in the left. To prove the converse let c be an element contained both in $\mathfrak{c}$ and $\mathfrak{a} \cup \mathfrak{b}$. Then $c = a + b$ or $b = -a + c$, hence b is contained in both $\mathfrak{c}$ and $\mathfrak{b}$.

From Theorem 1 and the general theory of Dedekind structures it follows that all the ordinary decomposition theorems will hold, in particular the theorems on the representation of an ideal as the direct union of direct indecomposable ideals, the representation as the union of irreducible ideals, the theorem of Jordan-Hölder, and the more general refinement theorem. It seems remarkable that these theorems can be deduced with such small assumptions on the system, since in particular there are no assumptions on the properties of the multiplication except the weak Distributive Law I.

It should be mentioned at this point that the ordinary laws of isomorphism do not hold in their regular form in generalized rings without further strong conditions of distributivity. If $\mathfrak{a}$ is an ideal then the additive quotient group $\mathfrak{R}/\mathfrak{a}$ exists and $\mathfrak{R}$ is homomorphic to $\mathfrak{R}/\mathfrak{a}$ with respect to addition, but the residue classes in $\mathfrak{R}/\mathfrak{a}$ do not form a generalized ring since no multiplication is defined. Similar remarks apply to the law of isomorphism

$$\mathfrak{a} \cup \mathfrak{b}/\mathfrak{a} \sim \mathfrak{b}/\mathfrak{a} \cap \mathfrak{b}$$

which holds only in the additive sense.

The laws may be obtained however when certain new distributive conditions are imposed. If $\mathfrak{a}$ is a left ideal, then $\mathfrak{R}$ is homomorphic to $\mathfrak{R}/\mathfrak{a}$ with the elements of $\mathfrak{R}$ as domain of left multipliers, provided:

If a, b, and c are arbitrary elements, then $a(b+c)=ab+c_1$, where c_1 belongs to the left ideal defined by c.

Similarly one finds that $\mathfrak{R}/\mathfrak{a}$ is a generalized ring to which $\mathfrak{R}$ is homomorphic when $\mathfrak{a}$ is a two-sided ideal, provided:

If a, b, and c are arbitrary elements, then

$$a(b+c)=ab+c_1, \qquad (b+c)a=ba+c_2$$

where c_1 and c_2 belong to the two-sided ideal defined by c.

Under the same conditions also the second law of isomorphism will hold in its ordinary formulation.

3. Further remarks. In ordinary ideal theory one is also often interested in *ideals with operators.* There exist certain sets of operators $A, B, \cdots$ for the ring such that each operator produces a new element $A:a$ from a given element a. As examples one may take automorphisms, differentiation, etc. For a generalized ring one can define similar operations. Usually one wishes to consider the ideals which are invariant with respect to these operations $A:\mathfrak{a}\subset\mathfrak{a}$, and one can prove under certain conditions that the structure of these ideals will also satisfy the Dedekind axiom. An analysis analogous to the preceding shows that this is the case provided the operations satisfy:

Distributive law for operators.

For any two elements a and b and any operator A one has $A:(a+b) = a_1+b_2$ where a and b belong to the (left, right) ideals defined by a and b.

In this connection one should mention that one may of course consider a generalized ring as a special case of a group with correspondences or operators. Those normal subgroups which are invariant with respect to a set of such operators will form a Dedekind structure if the operators satisfy the distributive law given above.

The generalized ring discussed above is not the most general case in which one can derive a set of ideals forming a Dedekind structure. For an arbitrary system with two operations certain conditions may be obtained for the rules of operation insuring such properties of the ideals.

We shall not go into details of this investigation, but only give the following sufficient conditions:

1. The system shall be a quasi-group with respect to addition, i. e., the equations $a+x=b$, $y+a=b$ shall have unique solutions.

2. An associative law for addition, $(a+b)+c=a+d$, where d belongs to the ideal $\{b, c\}$ defined by b and c.

3. The Distributive Law I for multiplication.

Of course the ideals have to be suitably defined as certain types of normal sub-quasi-groups. The Associative Law II is designed to insure the existence of co-set expansions in the additive quasi-group. These results may be derived by methods similar to those used in the study of quasi-groups by Hausmann and Ore.[2]

CHAPTER II. Products and residuals.

1. Products. From now on we shall suppose that the system $\mathfrak{R}$ under consideration is a generalized ring. If $a, b, c, \cdots$ is any set of elements of $\mathfrak{R}$ then we shall denote by $\{a, b, \cdots\}_l$, $\{a, b, \cdots\}_r$, $\{a, b, \cdots\}$ respectively the left, right and two-sided ideals generated by these elements. Any such ideal $\mathfrak{a}$ consists of all elements obtainable from the given ones by successive applications of the following three operations:

A) Addition and subtraction.

B) Transformation in the additive group.

C) Multiplication on the left (right, or both) by arbitrary elements.

Now let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals or even only two sets of elements. We then define their product as follows:

Product. The product $\mathfrak{a} \cdot \mathfrak{b}$ is the left (right, two-sided) ideal generated by all products $a \cdot b$ where $a \subset \mathfrak{a}$, $b \subset \mathfrak{b}$.

In the following we shall usually assume that $\mathfrak{a}$ and $\mathfrak{b}$ are left ideals and that the product $\mathfrak{a} \cdot \mathfrak{b}$ is their left product.

The preceding definition does not correspond to the one usually introduced in the ordinary rings. Here one defines:

Bilinear product. The product $\mathfrak{a} \cdot \mathfrak{b}$ of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ (or two arbitrary sets) is the set of elements of the form $c = \Sigma a_i b_i$ where $a_i \subset \mathfrak{a}$, $b_i \subset \mathfrak{b}$.

The obvious deficiency of this definition is that the resulting set is not an ideal except when one makes certain assumptions on the ring operations. If, however, the bilinear product is an ideal, it must be equal to the product as defined above.

The bilinear product is an additive group. To make it an ideal this group has to be normal and closed under the operation of multiplication on the left

---

[2] Hausmann and Ore, *loc. cit.* In a paper submitted to the *Bulletin of the American Mathematical Society* Mr. Murdoch has given a more detailed discussion of this case.

(right) by an arbitrary element of $\mathfrak{R}$. This leads us to the following two conditions:

Normality condition I. For any three elements a, b, c in the generalized ring one shall have $c + a \cdot b - c = \Sigma a_i b_i$ where the a_i belong to the ideal $\{a\}_l$ and the b_i to the ideal $\{b\}_l$.

Associative-distributive law I. For any two sets of elements $b_1, \cdots, b_n$ and $c_1, \cdots, c_n$ and an arbitrary a one has $a(\Sigma b_i c_i) = \Sigma b'_i c'_i$ where the b'_i belong to $\{b_1, \cdots, b_n\}_l$ and the c'_i to the ideal $\{c_1, \cdots, c_n\}_l$.

This last rather complicated law may be considered as a combination of an associative and a distributive law in the same way as the ordinary distributive and associative laws may be joined into the single relation

$$a(b_1c_1 + b_2c_2) = (ab_1)c_1 + (ab_2)c_2.$$

When $\mathfrak{a}$ is a left ideal and $\mathfrak{b}$ a right ideal the product $\mathfrak{a} \cdot \mathfrak{b}$ is seen to be a two-sided ideal if the normality condition and the associative-distributive law hold in a suitable left and right formulation.

We shall now indicate a few of the properties of the product. The following are direct consequences of the definition:

1. If $\mathfrak{b} \supset \mathfrak{c}$ then $\mathfrak{a} \cdot \mathfrak{b} \supset \mathfrak{a} \cdot \mathfrak{c}$, $\mathfrak{b} \cdot \mathfrak{a} \supset \mathfrak{c} \cdot \mathfrak{a}$.
2. $\mathfrak{a}(\mathfrak{b} \cap \mathfrak{c}) \subset \mathfrak{a} \cdot \mathfrak{b} \cap \mathfrak{a} \cdot \mathfrak{c}$ for any $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$.
3. If $\mathfrak{a}$ is a left ideal then $\mathfrak{b} \cdot \mathfrak{a} \subset \mathfrak{a}$.

We shall now turn to the fundamental

4. Distributive ideal relation.

$$(\mathfrak{a} \cup \mathfrak{b})\mathfrak{c} = \mathfrak{a} \cdot \mathfrak{c} \cup \mathfrak{b} \cdot \mathfrak{c}.$$

The proof of this relation requires conditions on the properties of the operations in the generalized ring, namely:

Distributive law II. Let a, b, c be arbitrary elements and d an element in the ideal $\{a, b\}_l$. Then the product $d \cdot c$ must belong to the ideal $\mathfrak{m} = \{\cdots, a_i c_i, \cdots, b_i c_i, \cdots\}_l$ where $a_i \subset \{a\}_l$, $b_i \subset \{b\}_l$, $c_i \subset \{c\}$.

If one assumes the special Distributive Law I such that the set of ideals becomes a Dedekind structure, then every element in $\{a, b\}_l$ is of the form $d = a_1 + b_1$ and the Distributive Law II becomes simply $(a + b)c \subset \mathfrak{m}$ where $\mathfrak{m}$ has the same meaning as before.

If one supposes that the generalized ring is so restricted that the ideal products may be defined as bilinear products, then the corresponding distributive law which is required for 4. takes the more explicit form:

Distributive law III. For any three elements a, b, c

$$(a+b)c = \Sigma a_i c_i + \Sigma b_i c'_i.$$

A direct consequence of 4. is:

5. For any two left ideals $\mathfrak{a}$ and $\mathfrak{b}$

$$(\mathfrak{a} \cup \mathfrak{b}) \cdot (\mathfrak{a} \cap \mathfrak{b}) \subset \mathfrak{a} \cdot \mathfrak{b} \cup \mathfrak{b} \cdot \mathfrak{a}.$$

One can also determine the conditions on the ring operations which insure:

6. Associative ideal multiplication.

$$(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c}).$$

Since the conditions are easily derived but slightly complicated in nature they may be omitted here. Another important law is:

7. Commutative ideal multiplication.

$$\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}.$$

If this law is to hold it implies that for any a and b, $b \cdot a \subset \{\cdots, a_i b_i, \cdots\}$ where $a_i \subset \{a\}_l$, $b_i \subset \{b\}_l$ and similarly for the product $a \cdot b$. When the ideal multiplication can be defined by bilinear products one must have $a \cdot b = \Sigma b_i a_i$, $b \cdot a = \Sigma a'_i b'_i$.

But in the case of the commutative law one can also use a different method and define a product which in all cases is commutative:

Commutative product. The commutative product of two ideals (sets) is

$$\mathfrak{a} \times \mathfrak{b} = \mathfrak{b} \times \mathfrak{a} = \mathfrak{a} \cdot \mathfrak{b} \cup \mathfrak{b} \cdot \mathfrak{a}.$$

To conclude let us say that the ring $\mathfrak{R}$ has a left ideal unit if $\mathfrak{R} \cdot \mathfrak{a} = \mathfrak{a}$ for all left ideals $\mathfrak{a}$. Furthermore we say as usual that two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are *relatively prime* if $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{R}$.

2. Two-sided ideals. For two-sided ideals some of the preceding properties may be specified further. It should be noted that the Distributive Law II is the only condition which is required to obtain these properties.

First one has the obvious:

1. For any two-sided ideals $\mathfrak{a}$ and $\mathfrak{b}$, $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$.

When this is applied to 5. in § 1 one finds

2. $(\mathfrak{a} \cup \mathfrak{b})(\mathfrak{a} \cap \mathfrak{b}) \subset \mathfrak{a} \cdot \mathfrak{b} \cup \mathfrak{b} \cdot \mathfrak{a} \subset \mathfrak{a} \cap \mathfrak{b}$.

From this relation follows immediately:

3. If $\mathfrak{a}$ and $\mathfrak{b}$ are two-sided relatively prime ideals and if $\mathfrak{R}$ is a left ideal unit, then $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$.

Let us say that a decomposition of an ideal in the form

$$\mathfrak{a} = \mathfrak{a}_1 \cup \mathfrak{a}_2 \cup \cdots \cup \mathfrak{a}_r$$

is *direct* when the cross-cut of each $\mathfrak{a}_i$ with the union of the remaining is the zero ideal. We can then prove:

THEOREM 1. *Let $\mathfrak{R}$ be a generalized ring with unit ideal and*

$$\mathfrak{R} = \mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_n = \mathfrak{b}_1 \cup \cdots \cup \mathfrak{b}_m \tag{1}$$

two direct decompositions of $\mathfrak{R}$ as the union of two-sided ideals. Then the two decompositions can be refined in such a way that they become identical.

Proof. We shall have to assume that the distributive ideal relation 4. holds both for left and right multiplication with two-sided ideals. One then finds

$$\mathfrak{b}_i = \mathfrak{b}_i\mathfrak{R} = \mathfrak{b}_i\mathfrak{a}_1 \cup \mathfrak{b}_i\mathfrak{a}_2 \cup \cdots \cup \mathfrak{b}_i\mathfrak{a}_n,$$
$$\mathfrak{a}_j = \mathfrak{R}\mathfrak{a}_j = \mathfrak{b}_1\mathfrak{a}_j \cup \mathfrak{b}_2\mathfrak{a}_j \cup \cdots \cup \mathfrak{b}_m\mathfrak{a}_j,$$

and both decompositions are obviously direct. When they are substituted in (1) the two decompositions become identical. It remains only to show that the resulting decompositions are direct, and this follows by repeated applications of the lemma:

4. *If* $\mathfrak{a} \cup \mathfrak{b} \cup \mathfrak{c} = \mathfrak{R}$ *and* $\mathfrak{b} \cap \mathfrak{c} = \{0\}$, $\mathfrak{a} \cap (\mathfrak{b} \cup \mathfrak{c}) = \{0\}$ *then*

$$\mathfrak{b} \cap (\mathfrak{a} \cup \mathfrak{c}) = \mathfrak{c} \cap (\mathfrak{a} \cup \mathfrak{b}) = \{0\}.$$

Proof. From the first condition together with 3. one finds

$$\mathfrak{b} \cap (\mathfrak{a} \cup \mathfrak{c}) = \mathfrak{b} \cdot (\mathfrak{a} \cup \mathfrak{c}) \cup (\mathfrak{a} \cup \mathfrak{c}) \cdot \mathfrak{b} = \{0\}.$$

Theorem 1 implies that the representation of $\mathfrak{R}$ as the direct union of direct indecomposable two-sided ideals is unique.

3. Residuals. We shall now turn to the definition of *residuals* or *ideal quotients* in a generalized ring. First let $\mathfrak{a}$ and $\mathfrak{b}$ be arbitrary subsets of $\mathfrak{R}$.

We define $\mathfrak{q}_l = (\mathfrak{a} : \mathfrak{b})_l = \mathfrak{b} \diagdown \mathfrak{a}$ as the set of elements q such that $q \cdot b \subset \mathfrak{a}$ for any b in $\mathfrak{b}$. We shall call $\mathfrak{q}_l$ the *left residual* of $\mathfrak{a}$ with respect to $\mathfrak{b}$. Similarly one defines the *right residual* $\mathfrak{q}_r = (\mathfrak{a} : \mathfrak{b})_r = \mathfrak{a}/\mathfrak{b}$. When $\mathfrak{a}$ and $\mathfrak{b}$ are sets consisting of a single element we obtain the ordinary quotients. Obviously the residual may be a void set in certain cases.

From now on we shall usually assume that $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of some kind and in order to insure that the residual is not void we shall impose the following rather natural condition: If the unit element of the additive group is denoted by 0 then $0 \cdot a = a \cdot 0 = 0$ for every a in $\mathfrak{R}$. This condition is equivalent to saying that the zero-ideal shall consist of the single element 0.

In the following let $\mathfrak{a}$ be a left ideal and $\mathfrak{b}$ a (left, right) ideal. Without any conditions on the operations in the generalized ring one can say little about the properties of the sets $\mathfrak{b} \diagdown \mathfrak{a}$ or $\mathfrak{a} \diagup \mathfrak{b}$. We shall now impose such conditions that the residual $\mathfrak{b} \diagdown \mathfrak{a}$ becomes a left ideal. Corresponding to the three properties characterizing an ideal one finds three conditions on the operations of the generalized ring. To simplify the formulation of these conditions, let a, b, and c denote arbitrary elements, while $\mathfrak{c}_a$ denotes a left ideal generated by products $a \cdot c_i$ where the c_i belong to the left ideal $\{c\}$.

THEOREM 2. *The residual $\mathfrak{b} \diagdown \mathfrak{a}$ where $\mathfrak{a}$ is a left ideal is itself a left ideal when the following three conditions hold:*

Normality condition II. $(a + b - a)c \subset \mathfrak{c}_b$.

Distributive law IV. $(a \pm b)c \subset \mathfrak{c}_a \vee \mathfrak{c}_b$.

Associative law I. $(ab)c \subset \mathfrak{c}_b$.

THEOREM 3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be left ideals. Then the residual $\mathfrak{b} \diagdown \mathfrak{a}$ is a two-sided ideal if also the following law is satisfied:*

Associative law II. $(ab)c \subset \mathfrak{c}_a$.

When the residual is shown to be an ideal one can substitute the equivalent definition in terms of ideal multiplication:

The residual $\mathfrak{q}_l = \mathfrak{b} \diagdown \mathfrak{a}$ is the largest ideal such that $\mathfrak{q}_l \cdot \mathfrak{b} \subset \mathfrak{a}$.

4. Properties of residuals. We shall now indicate some of the properties of residuals. We mention first the properties which hold for residuals of arbitrary sets with no assumptions on the ring operations.

1. $\mathfrak{R} \diagup \mathfrak{a} = \mathfrak{a} \diagdown \mathfrak{R} = \mathfrak{R}$ for any set $\mathfrak{a}$.

2. $\{0\} \setminus \mathfrak{a} = \mathfrak{R}$ if the set $\mathfrak{a}$ contains 0,
$\phantom{\{0\} \setminus \mathfrak{a}} =$ void set if $\mathfrak{a}$ does not contain 0.
3. If $\mathfrak{b} \supset \mathfrak{c}$ are any sets then $\mathfrak{b} \setminus \mathfrak{a} \subset \mathfrak{c} \setminus \mathfrak{a}$, $\mathfrak{a} \setminus \mathfrak{b} \supset \mathfrak{a} \setminus \mathfrak{c}$.
4. $\mathfrak{c} \setminus \mathfrak{a} \cap \mathfrak{b} = (\mathfrak{c} \setminus \mathfrak{a}) \cap (\mathfrak{c} \setminus \mathfrak{b})$
5. $\mathfrak{a} \cap \mathfrak{b} \setminus \mathfrak{c} \supset \mathfrak{a} \setminus \mathfrak{c}$, $\mathfrak{a} \cap \mathfrak{b} \setminus \mathfrak{c} \supset \mathfrak{b} \setminus \mathfrak{c}$.

We shall now turn to certain relations which combine the properties of left and right residuals.

6. For any sets $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ the relation $\mathfrak{a} / \mathfrak{b} \supset \mathfrak{c}$ implies $\mathfrak{c} \setminus \mathfrak{a} \supset \mathfrak{b}$.
7. $\mathfrak{a} / (\mathfrak{b} \setminus \mathfrak{a}) \supset \mathfrak{b}$, $(\mathfrak{a} / \mathfrak{b}) \setminus \mathfrak{a} \supset \mathfrak{b}$.
8. $\mathfrak{a} / ((\mathfrak{a} / \mathfrak{b}) \setminus \mathfrak{a}) = \mathfrak{a} / \mathfrak{b}$, $(\mathfrak{a} / (\mathfrak{b} \setminus \mathfrak{a})) \setminus \mathfrak{a} = \mathfrak{b} \setminus \mathfrak{a}$.

The last relation is a consequence of 7.

Let us now turn to the more special case where the sets in question are ideals. Then one sees immediately:

9. When $\mathfrak{a}$ is a left ideal then $\mathfrak{a} \setminus \mathfrak{a} = \mathfrak{R}$.
10. When $\mathfrak{b}$ is a left ideal then $\mathfrak{b} \setminus \mathfrak{a} = \mathfrak{b} \setminus \mathfrak{a} \cap \mathfrak{b}$.
11. For a left ideal $\mathfrak{a}$, $\mathfrak{b} \cap \mathfrak{c} \setminus \mathfrak{a} \supset (\mathfrak{b} \setminus \mathfrak{a}) \cup (\mathfrak{c} \setminus \mathfrak{a}.)$
12. When $\mathfrak{a}$ and $\mathfrak{b}$ are left ideals, $\mathfrak{c} \setminus \mathfrak{a} \cup \mathfrak{b} \supset (\mathfrak{c} \setminus \mathfrak{a}) \cup (\mathfrak{c} \setminus \mathfrak{b}.)$
13. If $\mathfrak{a}$ is a two-sided ideal $\mathfrak{b} \setminus \mathfrak{a} \supset \mathfrak{a}$, $\mathfrak{a} / \mathfrak{b} \supset \mathfrak{a}$.
14. If the generalized ring has an ideal unit then $\mathfrak{a} / \mathfrak{b} = \mathfrak{R}$ if and only if $\mathfrak{a} \supset \mathfrak{b}$.

When the Distributive Law II, hence the distributive ideal relation 4. § 1, holds, then one can obtain the two further properties:

15. Let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ be left ideals. Then $\mathfrak{b} \cup \mathfrak{c} \setminus \mathfrak{a} = (\mathfrak{b} \setminus \mathfrak{a}) \cap (\mathfrak{c} \setminus \mathfrak{a}.)$
16. $\mathfrak{b} \cup \mathfrak{a} \setminus \mathfrak{a} = \mathfrak{b} \setminus \mathfrak{a}$.

Finally one finds if ideal multiplication is associative

17. $(\mathfrak{c} \setminus \mathfrak{b}) \setminus \mathfrak{a} = \mathfrak{c}\mathfrak{b} \setminus \mathfrak{a}$.

We shall not go further into the properties of generalized rings. After the conditions for the principal ideal properties have been established one can proceed to define a prime ideal and prove certain decomposition properties much in the same way as in ordinary rings. An ideal is *nilpotent* if $\mathfrak{a}^n = \{0\}$ and the *radical* is the maximal nilpotent ideal. A generalized ring without radical is said to be *semi-simple*.[3]

---

[3] From this point on the investigations would follow the structural theory outlined by R. P. Dilworth: "Non-commutative residuated lattices," *Transactions of the American Mathematical Society*, vol. 46 (1939), pp. 426-444.

CHAPTER III. Applications to groups.

1. Commutator product. As an important application of the preceding theory we shall show that the ordinary theory of normal subgroups in a group may be considered as a special case of ideal theory in a generalized ring.

Let G be a group whose operations are written in additive form, and hence $a+b$ denotes the ordinary product and 0 the unit element. Any group may then be represented as a generalized ring when we define multiplication of two elements as their commutator, $a \circ b = a + b - a - b = b^a - b$, where $b^a = a + b - a$. This operation has various simple properties. Obviously $a \circ b = 0$ if and only if a and b commute additively. In particular $a \circ 0 = 0 \circ a = 0$, and $a \circ a = 0$. In an Abelian group every product vanishes. One sees immediately:

The ideals with respect to commutator multiplication are the normal subgroups and every ideal is two-sided.

Let us now turn to the properties of the ideal product $A \circ B$ of two ideals or normal subgroups A and B. This product consists of all elements generated by commutators $a \circ b$ and since the transform of any such commutator is of the form $a_1 \circ b_1$ it follows immediately that the product can be defined in the bilinear form. This may also be verified by an explicit formulation of the commutator rules in such a manner that the preceding laws are seen to be fulfilled.

From the relation $a \circ b + b \circ a = 0$ follows immediately:

Any two ideals are permutable, $A \circ B = B \circ A$.

It should be noted, however, that ideal multiplication is usually not associative.

For the commutator product one has the distributive laws

$$(a+b) \circ c = a \circ (b \circ c) + b \circ c + a \circ c,$$
$$c \circ (a+b) = c \circ a - (c \circ b) \circ a + c \circ b$$

or also

$$(a+b) \circ c = (b \circ c)^a + a \circ c, \qquad c \circ (a+b) = c \circ a + (c \circ b)^a.$$

These rules are sufficiently strong to imply all the preceding Distributive Laws I-IV and also the other distributive laws mentioned in § 2, Chapter I. As a consequence the distributive ideal relation 4. in § 1, Chapter II holds for multiplication on both sides and hence all the properties of ideal products are valid for normal subgroups.

The two identities $(a \circ b)^c = a^c \circ b^c$, $b^a \circ c = (b \circ c^{-a})^a$ show that the Normality conditions I and II are also satisfied. Similarly the relations

$$(a \circ b) \circ c = (b^a - b) \circ c = (a + (-a)^b) \circ c$$

will serve to verify the Associative laws I and II. At this point one might also mention the following identity

$$a \circ (b + c) + b \circ (c + a) + c \circ (a + b) = 0.$$

The preceding relations imply the existence of the residual $Q = A : B$ as the largest subgroup such that $Q \circ B \subset A$. This might of course have been established directly and one also finds: *The right and left residuals are equal.* All the results on residuals which have been obtained previously now apply automatically to residuals in groups.

Now let us turn to the intrinsic meaning of the product of two normal subgroups. The product $P = A \circ B$ consists of products of elements of the form $a \circ b$ and it is contained both in A and in B. If $P = 0$ the elements in the two groups commute. By applying this to the quotient groups one finds:

The product $P = A \circ B$ is the largest subgroup such that the elements of the two quotient groups A/P and B/P commute.

From the relation 2. in § 2, Chapter II follows $(A \cap B)^2 \subset A \circ B \subset A \cap B$, and hence: The product $A \circ B$ is contained in $A \cap B$ and contains the commutator group of $A \cap B$.

From the definition it follows that the residual $Q = A : B$ is the largest subgroup such that all the commutators of elements of Q with elements of B belong to A, and hence to $A \cap B$. This means that for any q in Q and b in B $qbq^{-1} = b \cdot d$ where d belongs to $A \cap B$.

The centre C of G may be defined as the maximal group such that $G \circ C = 0$. The necessary and sufficient condition that $G \circ A = A$ is that there exist no normal subgroup P in G such that A/P belongs to the centre of G/P. This implies that if there exist no normal subgroups A/B in G such that A/B belongs to the centre of G/B then G is an ideal unit and the previous theorem on the unique direct decomposition of the group into normal components will hold.

2. Other applications. On could also give the application of the preceding theory to groups by defining the product in an additive group as the transform $a \times b = b^a = a + b - a$. One finds the relations

$$a \times a = a, \quad a \times (-b) = -a \times b, \quad (a + b) \times c = a \times (b \times c)$$

and many others. From the point of view of ideal theory this generalized ring is not as interesting as the commutator ring. One finds that the normal subgroups are the left ideals while the only right ideal is the full group.

More interesting are the *Lie rings* with an Abelian additive group and the defining relations for multiplication

$$a \circ (b + c) = a \circ b + a \circ c, \quad (b + c) \circ a = b \circ a + c \circ a, \quad a \circ a = 0,$$
$$a \circ b + b \circ a = 0, \qquad a \circ (b \circ c) + b \circ (c \circ a) + c \circ (a \circ b) = 0.$$

From the point of view of ideal theory the Lie rings have about the same properties as the commutator ring of ordinary groups. The normality conditions and the distributive laws are trivially satisfied. By means of the last relation one finds that the associative-distributive law holds, and hence the ideals may be defined as a bilinear product. All ideals are seen to be two-sided and ideal multiplication is commutative. Similarly it follows that all residuals exist as two-sided ideals having all the properties previously derived. The only condition which is not satisfied is the associative law for ideal multiplication.

YALE UNIVERSITY.

SERIES EXPANSIONS IN LINEAR VECTOR SPACE.*

By ORRIN FRINK, JR.

1. Introduction. A familiar problem is that of associating with every element f of a function space F a series expansion $\sum_{n=1}^{\infty} \alpha_n p_n$, in terms of a fixed set $\{p_n\}$ of functions of F, the coefficients $\{\alpha_n\}$ being real or complex numbers. In order to include the more important function spaces as special cases, this problem is considered here for a real or complex Banach space. Many of the results, however, can be extended to linear vector spaces without a norm. The first question that arises is how to select the fixed elements $\{p_n\}$ so that the series expansion will be unique. It is clear that if uniqueness is expected, no element p_n should be a linear combination of the others, and it seems natural to impose the stronger condition that no element of $\{p_n\}$ be the limit of a sequence of linear combinations of the other elements of $\{p_n\}$. If this condition holds, the set $\{p_n\}$ is said to be *minimal*. It will be seen that this condition alone leads to an interesting theory of series expansions.

Kaczmarz and Steinhaus ([**2**], p. 264) have pointed out the connection between the property of minimality and biorthogonal systems. Hence it is not surprising to find that the expansion theory of this paper is just as general as the theory of biorthogonal expansions discussed by Banach ([**1**], p. 106) in the sense that a minimal set $\{p_n\}$ is always part of a biorthogonal system $\{p_n, f_n\}$. This fact, however, is a consequence of the expansion theory given here, and would be difficult to prove in any other manner in the general case. It will be seen that many properties of expansions in terms of a minimal set may be advantageously studied in terms of the property of minimality, without making any use of biorthogonality. The existence of a set of coefficient functionals $\{f_n\}$ which together with the minimal set $\{p_n\}$ form a biorthogonal system, is a necessary consequence of the uniqueness of the expansion.

Although no very sharp theorems of convergence or summability of series are to be expected in such a general situation, the method of determining the expansion coefficients used here suggests that *semiregular* methods of summability, hitherto neglected, are the natural methods to use with minimal series. Semiregular methods, though they may sum a series of numbers to the wrong sum, never assign the wrong sum to a minimal series.

Since the more usual special cases of biorthogonal series are so well

* Received April 18, 1940.

known, most of the applications given here are to series not usually treated as biorthogonal. In particular, the inclusion of complex Banach spaces allows of interesting applications to the theory of functions of a complex variable, including power series.

2. Definitions. A real or complex *Banach space* B is a linear vector space whose elements may be added together or multiplied by real or complex numbers subject to conditions found in Banach ([1], p. 26). To every element x of B is assigned a real number $|x|$ called the norm of x, so that (1) $|\theta| = 0$, where θ is the zero element of B, (2) $|x| > 0$ for $x \neq \theta$, (3) $|x + y| \leqq |x| + |y|$, and (4) $|\alpha x| = |\alpha| \cdot |x|$. Convergence of a sequence of elements to a limit is defined in terms of the distance $|x - y|$. A Banach space is complete; that is, every sequence having the Cauchy property converges. Elements of B will usually be denoted by Roman letters, and real and complex numbers by Greek letters.

The linear extension A^L of a set A of elements of B is the set of all (finite) linear combinations of elements of B. The closed linear extension A^C of A is the set of all *limits* of linear combinations of elements of A. The set A^C is the smallest Banach space which contains A and is closed in B.

A sequence P of elements $\{p_n\}$ is said to be *minimal* if no element of P is the limit of a sequence of linear combinations of the other elements of P, that is, if $p_n \bar{\epsilon} (P - p_n)^C$ for every n. An equivalent condition is that $(P - p_n)^C \neq P^C$ for every n.

3. Minimal series.

THEOREM 1. *If $P = \{p_n\}$ is a minimal sequence of elements of a real or complex Banach space, and $x \epsilon P^C$, then for every n there exists one and only one real or complex number ξ_n such that $(x - \xi_n p_n) \epsilon (P - p_n)^C$.*

It is no restriction to assume $n = 1$. Since $x \epsilon P^C$, a sequence $z_m \to x$ exists such that

$$z_m = \sum_{r=1}^{m} \alpha_{mr} p_r. \tag{1}$$

The sequence $\{\alpha_{m1}\}$ is bounded, for otherwise it would have a subsequence $\{\beta_k\}$ such that

$$\lim_{k\to\infty} \frac{1}{\beta_{k+1} - \beta_k} = 0. \tag{2}$$

Then denoting by $\{w_k\}$ the corresponding subsequence of $\{z_m\}$, it would follow that $v_k \to p_1$, where

$$v_k = p_1 - (w_{k+1} - w_k)/(\beta_{k+1} - \beta_k). \tag{3}$$

But the coefficient of p_1 in v_k is zero, which is contrary to the assumption that P is minimal. Hence $\{\alpha_{m1}\}$ is bounded, and a subsequence $\{\gamma_r\}$ can be selected from it which converges to some number ξ_1. Let the corresponding subsequence of $\{z_m\}$ be $\{u_r\}$. Then

$$u_r \to x, \text{ and } \gamma_r p_1 \to \xi_1 p_1.$$

Subtracting gives $u_r - \gamma_r p_1 \to x - \xi_1 p_1$. Since $u_r - \gamma_r p_1$ does not contain p_1, the number ξ_1 has the property required by the theorem. If ξ_1 were not unique, there would exist two sequences $\{s_n\}$ and $\{t_n\}$ of linear combinations of $(P - p_1)$ such that $s_n \to x - \xi_1 p_1$, and $t_n \to x - \eta_1 p_1$, where $\xi_1 \neq \eta_1$. Then $(s_n - t_n)/(\eta_1 - \xi_1) \to p_1$, contrary to the assumption that P is minimal. This completes the proof.

According to Theorem 1, if P is minimal every element x of P^C has assigned to it a unique sequence of numbers $\{\xi_n\}$. These are called the *expansion coefficients* of x in terms of P. The formal series $\sum_{n=1}^{\infty} \xi_n p_n$ formed with these coefficients is called the *series expansion* of x in terms of P. The relation between x and its series expansion is denoted by writing $x \sim \Sigma \xi_n p_n$. For n fixed, the dependence of ξ_n on x is denoted by $\xi_n = f_n(x)$. The functionals $f_n(x)$, $n = 1, 2, \cdots$, are called the *coefficient functionals* associated with the minimal set P.

THEOREM 2. *The coefficient functionals $f_n(x)$ associated with a minimal set P are additive, that is $f_n(\alpha x + \beta y) = \alpha f_n(x) + \beta f_n(y)$.*

This is an easy consequence of Theorem 1 and the definition of $f_n(x)$.

THEOREM 3. *If $P = \{p_n\}$ is minimal and $z_n \to x$,* where $x \epsilon P^C$ *and $z_n = \sum_{r=1}^{n} \alpha_{nr} p_r$, then, for every r, $\lim_{n\to\infty} \alpha_{nr} = \xi_r$, where $\xi_r = f_r(x)$ is the expansion coefficient of x in terms of P.*

The proof is similar to that of Theorem 1. The sequence $\{\alpha_{nr}\}$ for r fixed is bounded, and if any subsequence of it converges to a number α, then α has the property of ξ_r in Theorem 1. Since ξ_r is unique, any convergent subsequence of $\{\alpha_{nr}\}$ converges to ξ_r, hence the sequence itself converges to ξ_r.

Theorem 3 shows that all the expansion coefficients of x can be determined from any sequence of elements of P^L converging to x. As an illustration, from any sequence of polynomials converging uniformly to a real function $x(t)$, continuous on an interval $[a, b]$, can be determined the expansion coefficients of $x(t)$ in terms of any set of polynomials minimal on $[a, b]$. Such sets include all relatively orthogonal sets of polynomials for the interval $[a, b]$ or any smaller interval.

If $P = \{p_n\}$ is minimal, we shall denote by P_n the set $\{p_{n+1}, p_{n+2}, \cdots\}$, and if $x \epsilon P^C$, we shall denote by s_n the partial sum $\sum_{r=1}^{n} \xi_r p_r$ of the series expansion of x in terms of P.

THEOREM 4. *If P is minimal and $x \epsilon P^C$, then $x - s_n \epsilon (P_n)^C$. If z_n is any linear combination $\sum_{r=1}^{n} \alpha_r p_r$ of the elements $p_1, p_2, \cdots, p_n$ such that $z_n \neq s_n$, then $x - z_n \bar{\epsilon} (P_n)^C$.*

There exist numbers α_{mr} such that

$$\lim_{m\to\infty} \sum_{r=1}^{m} \alpha_{mr} p_r = x. \tag{4}$$

By Theorem 3,

$$\lim_{m\to\infty} \sum_{r=1}^{n} \alpha_{mr} p_r = \sum_{r=1}^{n} \xi_r p_r. \tag{5}$$

Subtracting (4) from (5) gives

$$\lim_{m\to\infty} \sum_{n+1}^{m} \alpha_{mr} p_r = x - \sum_{r=1}^{n} \xi_r p_r. \tag{6}$$

This proves the first part of the theorem. The second part follows from the uniqueness of the numbers ξ_n established in Theorem 1. Theorem 4 shows that in one sense, the partial sum s_n is the "best possible" approximation to x of all linear combinations of $p_1, p_2, \cdots, p_n$.

4. Normalization. Theorems 1 to 4 hold in linear vector spaces more general than Banach spaces, in which convergence is defined independently of the notion of the norm $|x|$ of an element x, since no use was made of the norm in the proofs. It was shown in Theorem 2 that the coefficient functionals are additive. To prove that they are also continuous, it is convenient to make use of the norm, and in particular to consider ways of normalizing the elements $\{p_n\}$ of a minimal set P. If $\alpha_n \neq 0$, the set $Q = \{\alpha_n p_n\}$ is also minimal if P is, and the theory of series expansions in terms of P and Q are equivalent, since if $x \sim \Sigma \xi_n p_n$, then $x \sim \Sigma (\xi_n/\alpha_n) \cdot \alpha_n p_n$. One way to normalize P would be to demand that $|p_n| = 1$ for every n. For certain purposes it is more convenient to define normalization in terms of the property of minimality, as follows.

Definition. If $P = \{p_n\}$ is minimal, then ρ_n is defined to be the minimum distance from p_n to the closed set $(P - p_n)^C$. Since p_n is not a member of the set $(P - p_n)^C$, this distance ρ_n is not zero. It can be seen that ρ_n is equal to the greatest lower bound of $|p_n - z|$ for all z in $(P - p_n)^L$.

Definition. The element p_n of the minimal set P is said to be ***normalized*** if $\rho_n = 1$. P is said to be *normalized* if all its elements are normalized. If P is minimal but not normalized, the equivalent set $Q = \{q_n\}$, where $q_n = p_n/\rho_n$, is normalized. It is interesting to note that any orthonormal basis $P = \{p_n\}$ for Hilbert space is normalized in the sense of the above definition.

THEOREM 5. *If $P = \{p_n\}$ is normalized and minimal, then* $|\sum_{r=1}^{n} \alpha_r p_r| \geqq |\alpha_k|$ *for* $1 \leqq k \leqq n$.

Suppose $|\sum_{r=1}^{n} \alpha_r p_r| < |\alpha_k|$. Then $\alpha_k \neq 0$, and dividing by α_k gives $|p_k - \sum_{r=1}^{n} \beta_r p_r| < 1$, where $\beta_k = 0$ and $\beta_r = -\alpha_r/\alpha_k$ for $r \neq k$. This contradicts the assumption that p_k was normalized.

THEOREM 6. *The coefficient functionals $\{f_n\}$ associated with a minimal set $P = \{p_n\}$ are continuous; that is, if $x_m \to x$, where x_m and x are in P^C, and $\xi_{mn} = f_n(x_m)$ and $\xi_n = f_n(x)$ are the n-th expansion coefficients of x_m and x in terms of P, then $\lim_{m\to\infty} \xi_{mn} = \xi_n$ for every n.*

In the proof we may assume that P is normalized and $n = 1$. Given $\epsilon > 0$, it is sufficient to show that there exists an N such that if $m > N$, then $|\xi_{m1} - \xi_1| < 3\epsilon$. By Theorem 1 there exist elements z_m and z which are linear combinations of $\{p_2, p_3, \cdots\}$ such that

$$|\xi_{m1} p_1 + z_m - x_m| < \epsilon, \qquad (7)$$
$$|\xi_1 p_1 + z - x| < \epsilon.$$

Since $x_m \to x$, there exists an N such that for $m > N$,

$$|x - x_m| < \epsilon. \qquad (8)$$

Combining (7) and (8) gives

$$|(\xi_{m1} - \xi_1) p_1 + w| < 3\epsilon, \qquad (9)$$

where $w = z_m - z$. However, by Theorem 5,

$$|(\xi_{m1} - \xi_1) p_1 + w| \geqq |\xi_{m1} - \xi_1|. \qquad (10)$$

Hence $|\xi_{m1} - \xi_1| < 3\epsilon$ for $m > N$, which was to be proved.

5. Biorthogonal systems. It has now been shown that the coefficient functionals $\{f_n\}$ associated with a minimal set $P = \{p_n\}$ are additive continuous functionals on the Banach space P^C. Such functionals are called *linear*, and they belong to the Banach space Q conjugate to P^C, consisting of all linear functionals on P^C. The norm $|f|$ of such a functional f is defined to be the least real number M such that $|f(x)| \leqq M|x|$ for all $x \epsilon P^C$

(Banach [1], p. 54). A set $\{p_n, f_n\}$ consisting of a sequence of elements $\{p_n\}$ and a sequence of functionals $\{f_n\}$ of a Banach space B is said to be a *biorthogonal system* if $f_m(p_n) = \delta_{mn}$. In terms of such a biorthogonal system every element x of B has the biorthogonal series expansion $x \sim \sum_{n=1}^{\infty} f_n(x) p_n$, and every linear functional f on B has the series expansion $f \sim \sum_{n=1}^{\infty} f(p_n) f_n$.

THEOREM 7. *If $P = \{p_n\}$ is minimal, and the associated coefficient functionals are $\{f_n\}$, then the set $\{p_n, f_n\}$ is a biorthogonal system, and the minimal series expansion of an element x of P^C in terms of P is the same as the biorthogonal expansion of x in terms of the system $\{p_n, f_n\}$. Conversely, if $\{p_n, f_n\}$ is a biorthogonal system, then the set $P = \{p_n\}$ is minimal.*

The first part of the theorem follows from Theorems 2, 3, and 6. To prove the second part, suppose $\{p_n, f_n\}$ is a biorthogonal system, but $P = \{p_n\}$ is not minimal. Then for some n, $\lim_{m\to\infty} z_m = p_n$, where z_m is a linear combination of elements of $(P - p_n)$. Since f_n is a continuous functional, $\lim_{m\to\infty} f_n(z_m) = f_n(p_n)$. But this is impossible, since $f_n(z_m) = 0$, and $f_n(p_n) = 1$.

Definition. The set of elements $P = \{p_n\}$ of a Banach space is said to be *weakly minimal* if no element p_n of P is the weak limit of a sequence of linear combinations of elements of $(P - p_n)$. The set of functionals $F = \{f_n\}$ on the space B is said to be *weakly minimal* if no functional f_n is the limit in the sense of weak convergence of functionals of a sequence of linear combinations of $(F - f_n)$.

THEOREM 8. *If $\{p_n, f_n\}$ is a biorthogonal system, then the sets $\{p_n\}$ and $\{f_n\}$ are both weakly minimal.*

The proof is similar to that of Theorem 7. Of course, if a set is weakly minimal it is necessarily minimal.

Since the space Q conjugate to P^C is also a Banach space, the theory of minimal expansions in Q is to some extent covered by the theory for P^C. There is the difficulty, however, that while P^C is separable, the conjugate space Q may not be separable. If Q is not separable, it is impossible for the linear combinations of the set $F = \{f_n\}$ to be dense in Q. In this case there are two possibilities. One may consider minimal expansions only for functionals f which are in the closed linear extension F^C. Or, since there is always a countable set which is weakly dense in Q (Banach [1], p. 124), one may consider biorthogonal systems $\{p_n, f_n\}$ such that the linear extension F^L is weakly dense in Q, where $F = \{f_n\}$, and Q is the space conjugate to P^C. Although with

weak convergence of functionals as the definition of limit, Q is not a Banach space, the theory of series expansions in Q in terms of a weakly minimal set is similar to that given here for Banach spaces.

Definition. If $x \epsilon P^C$, where $P = \{p_n\}$ is minimal, $\epsilon_n(x)$ is defined to be the greatest lower bound of $| x - z_n |$ for all linear combinations $z_n(x)$ of $\{p_1, p_2, \cdots, p_n\}$, and $\epsilon_n(x)$ will be called the nth *order of approximation of x by P.* It is clear that $\lim_{n\to\infty} \epsilon_n(x) = 0$, and that the bound $\epsilon_n(x)$ is actually attained for a least one z_n.

THEOREM 9. *If $P = \{p_n\}$ is minimal and normalized, and ξ_m is the mth expansion coefficient of $x \epsilon P^C$ in terms of P, then $| \xi_m | \leqq \epsilon_n(x)$ for all $m > n$.*

Proof. Let δ be any positive number, and let $m > n$. By definition of $\epsilon_n(x)$, there exist numbers α_r such that

$$(11) \qquad | x - \sum_{r=1}^{n} \alpha_r p_r | < \epsilon_n(x) + \delta.$$

By Theorem 4, there exists a linear combination z_m of the elements $p_{m+1}, p_{m+2}, \cdots$ such that

$$(12) \qquad | \sum_{r=1}^{m} \xi_r p_r + z_m - x | < \delta.$$

Adding (11) and (12) gives

$$(13) \qquad | \xi_m p_m + w_m | < \epsilon_n(x) + 2\delta,$$

where w_m is a linear combination of the elements $(P - p_m)$. Hence by Theorem 5, $| \xi_m | < \epsilon_n(x) + 2\delta$, since P is normalized. Since δ is arbitrary, it follows that $| \xi_m | \leqq \epsilon_n(x)$, which was to be proved.

THEOREM 10. *If $\sum_{n=1}^{\infty} \xi_n p_n$ is the series expansion of $x \epsilon P^C$ in terms of the minimal and normalized set $P = \{p_n\}$, then $\lim_{n\to\infty} \xi_n = 0$.*

This follows from Theorem 9 and the fact that $\epsilon_n(x) \to 0$. Theorem 10 is an analogue of the Riemann-Lebesgue theorem. Theorem 9 will be used later to derive a sufficient condition for the convergence of the series $\sum_{n=1}^{\infty} \xi_n p_n$ to x.

THEOREM 11. *If $\{f_n\}$ are the coefficient functionals associated with the minimal and normalized set $P = \{p_n\}$, then $| f_n | = 1$ for every n.*

Proof. The norm $| f_n |$ is defined to be the greatest lower bound of real numbers M_n such that

$$(14) \qquad | f_n(x) | \leqq M_n | x |$$

for all $x \in P^C$. Now the inequality (14) holds for $M_n = 1$, for let x be any element of P^C, and z be such that $| x - z | < \delta$ and $z = \sum_{r=1}^{m} \alpha_r p_r$, where $\alpha_n = \xi_n$. Such an element z exists by Theorem 1. Since P is normalized it follows from Theorem 5 that $| f_n(x) | = | \xi_n | \leqq | z | < | x | + \delta$. Since δ is arbitrary, $| f_n(x) | \leqq | x |$, which is (14) with M_n replaced by 1. On the other hand, (14) will not hold for all $x \in P^C$ if $M_n < 1$. For, since P is normalized, there exists for every $\delta > 0$ an element x of the form $p_n + \sum_{r \neq n} \alpha_r p_r$, such that $| x | < 1 + \delta$. It follows from (14) that $1 = | f_n(x) | \leqq M_n | x | < M_n(1 + \delta)$. Since δ is arbitrary, $M_n \geqq 1$. Hence $| f_n | = 1$.

6. Summability of minimal series. The connection between an element x and its minimal series expansion $\sum_{n=1}^{\infty} \xi_n p_n$ becomes more evident if it can be shown that the series is summable by some method to x. Theorems 1, 3 and 4, which show that the expansion coefficients ξ_n are limits approached by the coefficients of linear combinations of $\{p_n\}$ which actually converge to x, suggest the application of convergence factor methods of summability (C. N. Moore [**3**]) to minimal series. Such a method of summability is called *regular* if it always sums a convergent series of numbers to its actual sum. More general methods which are here called *semiregular* seem appropriate for minimal series in a function space or linear vector space. The summability theorems proved below also provide an answer to the question of whether or not a series $\sum_{n=1}^{\infty} \alpha_n p_n$ is the expansion of some element x of P^C.

The series (1) $\sum_{n=1}^{\infty} x_n$ of elements x_n of a real or complex Banach space B is said to be *summable to the element x of B by the method* (α_{mn}), if (α_{mn}) is a triangular matrix of real or complex numbers, where n ranges from 1 to m for $m = 1, 2, \cdots$, and $\lim_{m \to \infty} \sum_{n=1}^{m} \alpha_{mn} x_n = x$. The series (1) is said to be *summable to x by the method* (α_{mn}) *of infinite range*, if (α_{mn}) is an infinite matrix of real or complex numbers, where n ranges from 1 to ∞, provided $\sum_{n=1}^{\infty} \alpha_{mn} x_n$ converges for every m, and $\lim_{m \to \infty} \sum_{n=1}^{\infty} \alpha_{mn} x_n = x$.

A method of summability (α_{mn}) of finite or infinite range will be called *semiregular* if $\lim_{m \to \infty} \alpha_{mn} = 1$ for every n. A semiregular method which is not regular may sum a convergent series of numbers to a number different from the sum of the original series. In particular, any method of rearranging the order of the terms of a series is a semiregular method of summability. Such a method is in general not regular, as may be seen from the case of a condi-

tionally convergent series of numbers. The fact that rearranging the order of the elements $\{p_n\}$ of a minimal set P gives a set which is still minimal suggests the advisability of considering semiregular methods of summability of minimal series.

THEOREM 12. *If the series* (1) $\sum_{n=1}^{\infty} \xi_n p_n$ *is summable by a semiregular method* (α_{mn}) *of finite or infinite range to the element* x *of* P^C, *where* $P = \{p_n\}$ *is minimal, then* (1) *is the series expansion of* x *in terms of* P.

This follows from Theorems 3 and 6, and the definition of semiregular summability.

THEOREM 13. *If* (1) $\sum_{n=1}^{\infty} \xi_n p_n$ *is the series expansion of the element* x *of* P^C *in terms of the minimal set* $P = \{p_n\}$, *and if only a finite number of the coefficients* ξ_n *are zero, then* (1) *is summable to* x *by at least one semiregular method* (α_{mn}) *of finite range.*

Proof. Suppose $\xi_k \neq 0$ for $k > n$. By Theorem 4 there exists a sequence $z_m \to x$, of the form $z_m = s_n + \sum_{k=n+1}^{m} \beta_{mk} p_k$, where s_n is the n-th partial sum of (1) and $\lim_{m\to\infty} \beta_{mk} = \xi_k \neq 0$ for every $k > n$. Hence $z_m = \sum_{k=1}^{m} \alpha_{mk} \xi_k p_k$, where $\alpha_{mk} = 1$ for $k \leqq n$, and $\alpha_{mk} = \beta_{mk}/\xi_k$ for $k > n$. Since $\lim_{m\to\infty} \alpha_{mk} = 1$ for every k, the method (α_{mk}) is semiregular, which proves the theorem.

Since, without further assumptions concerning the minimal set P, it may happen that two different elements x and y of P^C have the same series expansion in terms of P, it follows from Theorem 13 that two different semiregular methods may sum the same series (1) to different limits x and y, while (1) may converge to still a third limit. An interesting example is the following. Consider the real Banach space C of real continuous functions $x(t)$ on the closed interval $[a, b]$, where $a < -1$ and $1 < b$. The norm $|x|$ of $x(t)$ is defined to be max $|x(t)|$ on $[a, b]$. The Legendre polynomials for the interval $[-1, 1]$ are minimal in C. Two functions $x(t)$ and $y(t)$ of C which differ only outside the interval $[-1, 1]$ have the same series expansion in terms of these polynomials. If this series expansion has only a finite number of zero coefficients, it is uniformly summable by different semiregular methods to the different functions $x(t)$ and $y(t)$.

It may also be seen that the hypothesis of Theorem 13 that only a finite number of the coefficients ξ_n are zero, cannot be entirely omitted. Consider a function $x(t)$ of the space C which is an even function on the interval $[-1, 1]$, but is not an even function on the larger interval $[a, b]$. All the

odd coefficients in the series expansion of $x(t)$ in terms of Legendre polynomials will vanish. Hence the series expansion of $x(t)$ is not summable to $x(t)$ by any semiregular method, since no such method can introduce odd polynomials not present in the original series, and a series of even polynomials can converge only to an even function.

7. Totality. If $P = \{p_n\}$ is minimal, the set of coefficient functionals $\{f_n\}$ associated with P is said to be *total* if the condition $f_n(x) = 0$ for all n implies that $x = \theta$ for x in P^C. (Banach [**1**], pp. 42, 106). A condition equivalent to totality is that the only element whose expansion coefficients are all zero, is the element θ.

THEOREM 14. *If the condition of totality holds for the minimal set P, then the elements x and y of P^C have the same series expansion in terms of P only if $x = y$.*

THEOREM 15. *If the condition of totality holds for the minimal set $P = \{p_n\}$, and the series expansion $\sum_{n=1}^{\infty} \xi_n p_n$ of an element x of P^C in terms of P is summable by a semiregular method of finite or infinite range to the element y, then $x = y$.*

This is a consequence of Theorems 12 and 14. Together with Theorem 12, Theorem 15 insures that semiregular summability methods will never sum a minimal series to the wrong sum. Together with Theorem 13, it provides a necessary and sufficient condition that a series of the form $\sum_{n=1}^{\infty} \alpha_n p_n$ be the minimal series expansion of some element x of P^C. Since convergence is a special case of semiregular summability, it follows from Theorem 15 that if totality is assumed, the series expansion of an element x, if it converges, must converge to x. (Banach [**1**], p. 106).

8. Absolute convergence. A series $\sum_{n=1}^{\infty} x_n$ of elements of a real or complex Banach space B is said to be *absolutely convergent* if the series of norms $\sum_{n=1}^{\infty} |x_n|$ converges. Since a Banach space is complete, absolute convergence implies the existence of a sum x of the series.

THEOREM 16. *If $P = \{p_n\}$ is minimal and normalized, and the condition of totality holds, and $\epsilon_n(x)$ is the order of approximation by P of the element x of P^C, and if the series of numbers $\sum_{n=1}^{\infty} \epsilon_n(x) \mid p_n \mid$ converges, then the series expansion $\sum_{n=1}^{\infty} \xi_n p_n$ of x in terms of P converges absolutely to x.*

This follows from Theorems 9 and 15. Absolute convergence of a series in a Banach space implies various types of unconditional convergence (Banach [1], p. 240).

9. Density. The condition of totality is related to the question of whether the linear combinations of the coefficient functionals $\{f_n\}$ associated with a minimal set P are weakly dense in the space Q conjugate to P^C.

THEOREM 17. *If the linear combinations of the coefficient functionals $\{f_n\}$ associated with a minimal set $P = \{p_n\}$ are weakly dense in the space of functionals Q conjugate to P^C, then the set $\{f_n\}$ is total.*

Suppose $\{f_n\}$ is not total, but that on the contrary $f_n(x) = 0$ for all n, for some element x of P^C not equal to θ. Then there exists a functional f in Q such that $f(x) = 1$. Since linear combinations of $\{f_n\}$ are dense in Q, there exists a sequence $\{g_n\}$ of such linear combinations converging to f in the sense of weak convergence of functionals. But $g_n(x) = 0$ for all n, whereas $f(x) = 1$, which is a contradiction.

The case in which the coefficient functionals $\{f_n\}$ are weakly dense in the space Q conjugate to P^C is an important one in the applications of biorthogonal series, since biorthogonal expansions then exist for all elements of Q as well as of P^C. Hence it is of interest that the condition of totality holds in this case.

10. Biorthogonalization. Let B be a separable real or complex Banach space, and $\bar{B}$ be its conjugate. Let the sequences $\{q_n\}$ and $\{h_n\}$, each linearly independent, be given such that their linear combinations are respectively dense and weakly dense in B and $\bar{B}$, and suppose that $h_n(q_n) \neq 0$. Then there exists a procedure for constructing a biorthogonal system $\{p_n, f_n\}$ of linear combinations of $\{q_n\}$ and $\{h_n\}$. (Kaczmarz and Steinhaus [2], p. 265). Let $p_1 = q_1$ and $f_1 = \beta_{11}h_1$ so that $f_1(p_1) = 1$. Similarly, let $p_2 = \alpha_{21}q_1 + q_2$, and $f_2 = \beta_{21}h_1 + \beta_{22}h_2$ so that $f_1(p_2) = f_2(p_1) = 0$; $f_2(p_2) = 1$, and in general, $p_n = \sum_{r=1}^{n-1} \alpha_{nr}q_r + q_n$, $f_n = \sum_{r=1}^{n} \beta_{nr}h_r$, so that $f_m(p_n) = f_n(p_m) = 0$ for $m < n$, while $f_n(p_n) = 1$. The conditions are sufficient to determine the coefficients α_{nr} and β_{nr}, and it can be seen that $\beta_{nn} \neq 0$. The elements $\{p_n\}$ are minimal by Theorem 7 and the set $\{f_n\}$ is total by Theorem 17.

11. Schauder series. A particularly simple case of minimal series is the theory of series expansions in terms of the functions defined by Schauder (Kaczmarz and Steinhaus [2], p. 50). Let C be the space of real continuous functions $x(t)$ defined on the interval $[a, b]$, the norm of $x(t)$ being max

7

$|x(t)|$ on $[a, b]$. Let $\{t_n\}$ be a sequence of distinct points dense in $[a, b]$ with $t_1 = a$ and $t_2 = b$. Define $p_1(t)$ and $p_2(t)$ to be linear on $[a, b]$ with $p_1(t_1) = p_2(t_2) = 1$ and $p_1(t_2) = p_2(t_1) = 0$. For $n > 2$, let $p_n(t)$ be continuous on $[a, b]$, and linear on the intervals into which the points $\{t_1, t_2, \cdots, t_n\}$ divide $[a, b]$, with $p_n(t_n) = 1$ and $p_n(t_k) = 0$ for $k < n$. The set $P = \{p_n(t)\}$ is minimal in C, and the linear combinations of elements of P, being arbitrary polygonal lines with corners at the points $\{t_n\}$, are dense in C. The minimal series expansion of an arbitrary element $x(t)$ of C in terms of Schauder functions converges uniformly to $x(t)$, so that these functions form a *basis* for the space C. The coefficient functionals $\{f_n(x)\}$ depend only on the value of $x(t)$ at three points, and may be expressed in terms of the second divided difference $[x_i, x_n, x_j]$, where t_i and t_j are the points of $\{t_1, t_2, \cdots, t_n\}$ immediately to the left and right of t_n. Since the linear combinations of these coefficient functionals $\{f_n\}$ are weakly dense in the space $\bar{C}$ conjugate to C, an arbitrary functional of $\bar{C}$ may be expanded in a minimal series in terms of the set $\{f_n\}$.

Kaczmarz and Steinhaus state incorrectly ([**2**] p. 51) that any finite number of Schauder functions $\{p_n(t)\}$ for $n > 2$ may be omitted, and the remaining functions will still have their linear combinations dense in C. If this were true, the Schauder functions would not be minimal. Actually, they are part of the biorthogonal system $\{p_n(t), f_n(x)\}$ and hence minimal by Theorem 7. If a finite number of the points $\{t_n\}$ are omitted, then infinitely many of the Schauder functions must be redefined.

12. Minimal polynomials. A method of constructing sets of minimal polynomials $\{p_n(t)\}$, where $p_n(t)$ is of degree exactly $n - 1$, is the following. In the space C of real functions $x(t)$ continuous on the interval $[a, b]$, with norm defined to be max $|x(t)|$ on $[a, b]$, let $\{h_n(x)\}$ be any set of independent linear functionals on C whose linear combinations are weakly dense in $\bar{C}$. If $x_n(t)$ is the function equal to t^{n-1} on $[a, b]$, the process of biorthogonalization applied to the sets $\{x_n(t)\}$ and $\{h_n(x)\}$ leads to a minimal set $\{p_n\}$ of polynomials of the form $p_n(t) = t^{n-1} + \sum_{r=1}^{n-1} \alpha_{nr} t^{r-1}$. The associated coefficient functionals $\{f_n\}$ are linear combinations of $\{h_n\}$, and their linear combinations are weakly dense on $\bar{C}$. Hence the set $\{f_n\}$ is total by Theorem 17. In particular, if $\{t_n\}$ is a sequence of distinct points dense on the interval $[a, b]$, the functionals $h_n(x) = x(t_n)$ have their linear combinations weakly dense in the space $\bar{C}$. The corresponding minimal polynomials obtained by biorthogonalization are the Newton polynomials $\{p_n(t)\}$ where $p_1(t) \equiv 1$, and $p_n(t) =$

$\prod_{r=1}^{n-1}(x - t_r)$ for $n > 1$. The associated coefficient functionals are $f_1(x) = x(t_1)$, and $f_n(x) = [x_1, x_2, \cdots, x_n]$, the divided difference operation, for $n > 1$. Thus Newton interpolation series are a special case of minimal polynomial series. (Walsh [**4**], p. 53). Applying Theorems 12, 13, and 15 gives

THEOREM 18. *If $x(t)$ is a function real and continuous on the interval $[a, b]$, and $\{t_n\}$ is a sequence of distinct points dense on $[a, b]$, then*

1) *if the Newton interpolation series for $x(t)$ in the points $\{t_n\}$ is uniformly summable by a semiregular method of finite or infinite range to a function $y(t)$, then $x(t) = y(t)$, and*

2) *if only a finite number of coefficients of the Newton interpolation series for $x(t)$ are zero, then the series is uniformly summable to $x(t)$ by at least one semiregular method of finite range.*

13. Applications to complex variable theory. There are several types of complex Banach space which correspond to the space C of real continuous functions. For example, let K be a Jordan arc in the finite complex plane. In the first place there is the Banach space C_k of all complex valued continuous functions $f(t)$ defined on K, with the norm of $f(t)$ defined to be max $|f(t)|$ on K. Then there is the space A_K of functions analytic on K, with the same definition of norm. The question of the form of the most general linear functional on these spaces arises when the conjugate spaces are considered. The theory is different depending on whether K is assumed to be rectifiable or not.

Again, let B be a simple closed curve in the finite complex plane, and let G be the interior of B. Consider the Banach space $\mathscr{C}_B$ of all functions $f(z)$ holomorphic in G and continuous in $G + B$, with the norm of $f(z)$ defined to be max $|f(z)|$ on B. Again the question of the form of the most general linear functional on the space $\mathscr{C}_B$ arises, and again the treatment differs depending on whether B is rectifiable or not. In either case, however, Walsh has shown ([**4**] pp. 36, 39) that the set of all polynomials is dense in all three spaces C_K, A_K, and $\mathscr{C}_B$. Hence in all of these spaces there is a theory of the series expansion of a function in terms of a set of minimal polynomials. This theory includes as a special case the theory of expansions in a series of polynomials relatively orthogonal on the arc K or the curve B with respect to a weight function. It also includes the theory of polynomial interpolation series. In each case the results of this paper concerning semiregular summability apply, and lead to theorems of the type of Theorem 18.

In all three Banach spaces there is the theory of Newton interpolation series in a set of distinct points $\{t_n\}$ everywhere dense on the arc K or the

curve B. In the space A_K there is also the theory of interpolation in a set of points not necessarily distinct or dense in K. In the space $\mathcal{C}_B$ there is the theory of expansions in polynomial interpolation series in points of the interior G of the curve B not necessarily distinct (Walsh [**4**]). In particular, power series are of this type.

THEOREM 19. *If the function $f(z)$ is holomorphic on the interior G of a simple closed curve B of the finite z-plane, and $f(z)$ is continuous on $G+B$, and if the power series expansion* (1) $\sum_{n=0}^{\infty} f^{(n)}(a)(z-a)^n/n!$ *of $f(z)$ about the point a of G is uniformly summable by a semiregular method of finite or infinite range to a function $g(z)$ on B, then $f(z)=g(z)$. Furthermore, if only a finite number of the coefficients of* (1) *are zero, the series* (1) *is uniformly summable to $f(z)$ on $\bar{G}$ by at least one semiregular method of finite range.*

This follows from Theorems 12, 13, and 15, and from the fact that polynomials are dense on the space $\mathcal{C}_B$, that the system $\{(z-a)^n, f^{(n)}(a)/n!\}$ is biorthogonal, and that the functionals $\{f^{(n)}(a)/n!\}$ are total.

Of course there are many other types of series expansion in the theory of functions of a complex variable to which the theory of minimal series of this paper can be applied. From the few examples given here it can be seen that the value of treating such series expansions from the point of view of Banach spaces is chiefly that this viewpoint suggests new problems and methods, and links together subjects that might otherwise seem unrelated. In the case of any one type of expansion it is to be expected that special methods will give sharper results than any to be derived from the general theory given here.

THE PENNSYLVANIA STATE COLLEGE.

BIBLIOGRAPHY.

1. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.
2. S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Warsaw, 1935.
3. C. N. Moore, *Summable series and convergence factors*, Colloquium Publications 22, New York, 1938.
4. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Colloquium Publications 20, New York, 1935.

ON LEIBNIZ'S DEFINITION OF PLANES.*

By Herbert Busemann.

1. The well known unsatisfactory Euclidean definitions of straight lines and planes have evoked early attempts to replace them by better ones. Leibniz [1] proposed *to define a plane as the locus of points which have equal distances from two given points,* and the straight line as locus of points having equal distance from three given points. Since the idea of metric space or anything which could be substituted for it had not yet been developed, this attempt of Leibnz did not grow into a well-founded theory. The present note tries to give a critical investigation of the implications of Leibniz's definition from the point of view of metric spaces.

Once we have a metric, the straight lines must be the geodesics. To furnish something which deserves the name of a plane the requirement has to be added to Leibniz's definition that *the locus of those points which have equal distances from two given points is a full geodetic manifold,* i. e. that with any two points it should also contain the geodesics connecting them. Our purpose is to show that *this condition essentially already implies the Euclidean or Hyperbolic character of the metric.*

We shall see first (section 2) that in order to exclude certain degenerate metric spaces one has to assume the space to be *convex, externally convex and finitely compact.* Let these conditions be satisfied and assume *furthermore that for any two points A, B the locus $m(A, B)$ of those points which have equal distances from A and B with any two points X, Y also contains each geodesic through X and Y.* Then we shall see in sections 3 and 4 that *the space is homeomorphic to the Euclidean space E_n of some dimension n and that its metric is Euclidean or Hyperbolic.* It is not sufficient to require that each segment connecting x and y is contained in $m(A, B)$ unless the condition of external convexity is replaced by strict external convexity.

A motion of a metric space is a single valued mapping of the space onto itself which preserves distances. The Euclidean or Hyperbolic geometries have the property of free movability in this sense: if the two sets σ and σ' are congruent then a motion of the whole space exists carrying σ into σ'. It is obvious that under certain convexity and compactness conditions the Euclidean

* Received April 29, 1940.

[1] *Mathematische Schriften*, zweite Abteilung 1, p. 166. Later, many other mathematicians took up Leibniz's definition. For references see F. Enriques, "Prinzipien der Geometrie," *Enzyklopädie der Math. Wiss.*, vol. III 1 (1907), p. 19.

and Hyperbolic geometries are the only ones with free movability. The question arises whether the nature of the sets σ can be restricted.

One easily derives from our theorem:

Let an internally and externally strictly convex, finitely compact metric space have the property that to any two congruent triples ABC and $A'B'C'$ a motion exists carrying ABC into $A'B'C'$. Then its metric is either Euclidean or Hyperbolic.

This theorem does not claim to be final, since probably further reductions of the hypothesis will be possible.

2. A very simple example of a finite geometry in which Leibniz's axioms hold can be gotten as follows. Let the space consist of $n+2$ points $P_1 \cdots P_{n+2}$ and put $P_iP_j = 1$ for $i \neq j$, $P_iP_i = 0$. The locus of those points which have equal distances from r given points will consist of the $n+2-r$ points different from the r given points. Calling any set of m points an $(m-1)$-dimensional linear space L_{m-1}, we see that exactly one L_{m-1} passes through any m points, which are in no L_{m-2}, and that an L_{m-1} is the locus of those points which have equal distances from $n+2-m$ fixed points.

This trivial example shows that some assumption regarding connectivity is necessary, *but even the strongest assumptions in this repect without convexity would not be sufficient to insure the Euclidean or Hyperbolic character of the metric.* For let $e(A,B)$ be the Euclidean metric of the E_n. Putting

$$AB = e(A,B) + \log(1 + e(A,B))$$

we have $AB + CB > AB$ unless $C = A$ or $C = B$. The geodesics of the metric are still the Euclidean straight lines (for "straight line" we shall use the abbreviation "s. l."). The loci $m(A,B)$ are the hyperplanes, which therefore with any two points also contain the geodesics connecting them. We therefore require *convexity*, i. e. *that to any two points x, z a point y different from x and z exists with*

$$xy + yz = xz.$$

(If this relation holds for three different points x, y, z, we shall write (xyz)).

Every convex subset of the E_n satisfies this condition. Therefore *external convexity* must be required, i. e. *that with any two points x, y also points v and w with (vxy) and (xyw) exist.*

But this would still admit open convex sets in E_n and also certain denumerable subsets of E_n; to exclude them we have to require *completeness.* In order to also exclude spaces of infinite dimension we require the stronger

property of *finite compactness,* or the validity of the Bolzano-Weierstrass principle: *every bounded sequence of points has an accumulation point.*

Call *"straight line"* (*s. l.*) any subset of our space which is congruent to a Euclidean straight line. It was proved by Menger [2] that under the above conditions an *s. l.* passes through any two given points. In particular there are segments connecting any two given points.

We now require for each pair of points $K \neq L$:

L_1: *If* $m(K, L)$ *contains* A, B *then it also contains each segment connecting* A *and* B.

Then the segment connecting two points is unique. For otherwise two points R and S could be found and two segments s_1 and s_2 connecting them, which have only the points R and S in common. Let C_1 and C_2 be the midpoints of R and S on s_1 and s_2 respectively. Then $m(C_1, C_2)$ would contain R and S, therefore s_1 and with it C_1. But C_1 cannot be on $m(C_1, C_2)$ since $C_1 \neq C_2$. We shall designate by $\overline{RS}$ the unique segment from R to S.

But *it does not follow from* L_1 *that the whole s. l.* g *through* A, B *is in* $m(K, L)$ *as soon as the points* A *and* B *are on* $m(K, L)$, as the following example shows: Take three rays in E_2 issuing from the point O. For any two points on the same ray we put AB equal to the Euclidean distance. If A, B are on different rays we put $AB = AO + OB$. For any two points K, L which have not the same distance from O, the set $m(K, L)$ consists of one point. If $KO = OL$, the set $m(K, L)$ will consist of the ray which does not contain K or L; hence $m(K, L)$ does not contain the $s. l.$ connecting two different points of $m(K, L)$. Therefore we must either require that the external convexity is strict:

s_0. *For any two points* x, y *and any number* $r > 0$ *there exists at most one point such that* (xyw) *and* $yw = r$,

or we have to replace L_1 by the stronger condition

L_2: *With any two points* A, B, $A \neq B$, *each s. l. through* A, B *is contained in* $m(K, L.)$.

Our main theorem will then be

THEOREM 1. *A convex, externally convex, finitely compact metric space, in which conditions* s_0 *and* L_1 *or in which condition* L_2 *is satisfied, is congruent to the Euclidean or Hyperbolic space of some finite dimension.*

[2] "Untersuchungen über allgemeine Metrik," *Mathematische Annalen,* vol. 100 (1928), pp. 73-163, in particular pp. 87 ss.

Since the segment $\overline{AB}$ is unique it follows from s_0 immediately that the $s.\,l.$ connecting A and B (if $A \neq B$) is unique. We designate it by $\underline{AB}$. The set consisting of A, B and those points X for which (ABX) or (AXB) we designate by $\underrightarrow{AB}$. s_0 is an immediate consequence of L_2. For if s_0 was not true, four different points A, B, C_1, C_2 would exist with the properties (ABC_1), (ABC_2) and $BC_1 = BC_2$. Then also $AC_1 = AC_2$, hence $m(C_1, C_2)$ would contain each $s.\,l.$ through A and B. Since segments are unique, each $s.\,l.$ h through A and C_1 must contain B, hence h would also be an $s.\,l.$ through A and B and therefore contained in $m(C_1, C_2)$, C_1 is not in $m(C_1, C_2)$.

We see that L_2 contains s_0 and L_1. We shall show now that s_0 and L_1 imply L_2.

Assume for an indirect proof that $m(K, L)$ contains two points A, B, but that there is a point Y_0 on $\underline{AB}$ which does not belong to $m(K, L)$, for instance $Y_0K < Y_0L$. Y_0 cannot belong to $\overline{AB}$ on account of L_1. Let X be any point of $\overline{BL}$, and Y the point on $\underrightarrow{AX}$ for which

$$AY = AX + BY_0 \frac{XL}{BL}.$$

For $X = B$ we have $Y = Y_0$ and therefore $KY < LY$, for $X = L$ we have $Y = L$ and therefore $KY > LY$. As X traverses $\overline{BL}$, the point Y traverses a Jordan arc from Y_0 to L, therefore X must pass a point X_1, such that for the corresponding point Y, we have $KY_1 = LY_1$, or $Y_1 \subset m(K, L)$. On account of L_1 the segment $\overline{Y_1A}$ belongs to $m(K, L)$. Hence X would be on $m(K, L)$ but

$$LX_1 = LB - BX_1 = KB - BX_1 < KX_1.$$

We see that both sets of conditions in Theorem 1 are equivalent and imply:

μ: *The space is metric and finitely compact; any two different points determine uniquely a s. l. which passes through them. For any* $K \neq L$ *the locus* $m(K, L)$ *contains with any two points the whole s. l. connecting them.*

To have a short expression we call a space satisfying this condition μ a μ-space.

3. We shall prove first that μ-spaces are linear, in the sense expressed by theorems (d), (e) (g) of this section.

A point F of a set σ is called a foot of the point P if PF = greatest lower bound of PX as X varies over σ. The center R of KL is the only foot of K (or L) on $m(K, L)$. For if X is any point on $m(K, L)$ we have

$$2KX = KX + XL > KL = 2KR.$$

In particular R is the only foot of K (or L) on every $s.\,l.$ g in $m(K,L)$ through R. Let $\bar{X} \neq R$ be any point of g. As X traverses the ray of g opposite to $\overrightarrow{R\bar{X}}$, the distance KX' changes continuously from KR to ∞ and therefore X' traverses a position $\bar{X}'$ for which $\bar{X}'K = \bar{X}K$. Since $g \subset m(K,L)$ we have

$$\bar{X}L = \bar{X}K = \bar{X}'K = \bar{X}'L.$$

Hence K and L are in $m(X,X')$ and $\underline{KL}$ will be contained in $m(X,X')$. Since $R \subset \underline{KL}$ we have $\bar{X}R = \bar{X}'R$. We have found

(a) *If g is any straight line in $m(K,L)$ through R (R is the center of KL) and if $\bar{X}$, $\bar{X}'$ are points on g with $K\bar{X} = K\bar{X}'$ then $\bar{X} = \bar{X}'R$.*

We apply (a) to $\underline{KL}$ as $s.\,l.$ in $m(X,X')$. Taking any two points K' and L' on $\overrightarrow{RK}$ and $\overrightarrow{RL}$ respectively and with $RK' = RL'$ we see that $\bar{X}K' = \bar{X}L'$, so X is a point of $m(K',L')$, and $\bar{X}$ being arbitrary, we conclude that $m(K',L') \subset m(K,L)$. In the same way one gets $m(K,L) \subset m(K',L')$.

(b) *Let K and L be any two different points, R their center. If K' and L' are any two different points on $\underline{KL}$, which have R as center, one has $m(K,L) = m(K',L')$.*

Therefore $m(K,L)$ only depends on the $s.\,l.$ h which carries K and L, and on the point R. We shall also use the notation $m(h,R)$ instead of $m(K,L)$ or $m(K',L')$.

If all points of an $s.\,l.$ have the point F as only foot in the set σ, we call h a perpendicular to σ. Since K has R as only foot on $m(R,h)$ and $m(K,L) = m(K',L')$, K' will have R as only foot on $m(R,h)$ and on every $s.\,l.$ in $m(R,L)$ through R, hence h is perpendicular to $m(R,h)$ and to every $s.\,l.$ in $m(R,h)$ through R. With the same notations as above we see that g is perpendicular to h because $h = \underline{KL} \subset m(\bar{X},\bar{X}')$. We shall prove now

(c) *As R traverses h the sets $m(R,h)$ cover the space simply.*

Let X be any point not on h, F a foot of X on h, and Q any point on L with $XQ > XF$. On the ray of h opposite to $\overrightarrow{FQ}$, we can find a point Q' with $KQ' = XQ$. Therefore $X \subset m(Q',Q)$; this shows that the sets $m(R,h)$ cover the space. Let F' be the center of $\overline{QQ'}$; according to the preceding considerations, $\underline{XF'}$ is perpendicular to h, therefore F' is the only foot of X on g and $F' = R$, or $m(Q,Q') = m(F,h)$. There can be no surface $m(\bar{F},h)$ with $\bar{F} \neq F$ through X, because $\bar{F}$ would be another foot of X on L.

(d) *An n-dimensional μ-space is homeomorphic to E_n.*

Proof. The theorem is true for 1-dimensional μ-spaces. Assume it to be true for s-dimensional μ-spaces, $s < n$, and fix an *s. l.* h. For $R \subset h$, the set $m(R, h)$ is a μ-space; for if K, L are any two points in $m(R, h)$, the set $m(K, L)$ of those points in $m(R, h)$ which have equal distances from K and L is simply the set $m(K, L) \cdot m(R, h)$, which also satisfies condition μ. Let K, L be any two points on h with R as center. Each KX with $X \subset m(K, L) = m(R, h)$ intersects $m(R, h)$ only at X, otherwise $\overrightarrow{KX}$ would be contained in $m(K, L)$. Let Y be the point on $\overline{KX}$ with $YX = KR$. The segments $\overline{YX}$, $X \subset m(R, h)$ form, topologically, the product π of $m(R, h)$ and a segment, hence [3]

$$\dim \pi = \dim m(R, h) + 1$$

or $s = \dim m(R, h) \leq n - 1$. Therefore $m(R, h)$ is homeomorphic to the E_s. It follows from (c) that the whole space is homeomorphic to E_{s+1}, therefore $s = n - 1$.

A finitely compact metric space of dimension d, which with any two points also contains exactly one *s. l.* connecting them, will be called a *d-dimensional linear space* L_d. We shall prove:

(e) *A d-dimensional linear subspace L_d of an n-dimensional μ-space is itself a μ space. A set $m(X, Y)$ either contains L_d or is disjoint from L_d or intersects it in a $(d-1)$-dimensional μ-space.*

Let K, L be any two points in L_d. Then $m(K, L) \cdot L_d$ is the set of all points in L_d which have equal distances from K and L. Since $m(K, L)$ and L_d both with any two points also contain the *s. l.* through them, $m(K, L) \cdot L_d$ does, hence L_d is a μ-space. Let now X, Y be any two points in the whole space. If $m(X, Y)$ does not contain L_d and is not disjoint from L_d, its intersection with L_d is a linear subspace L of L_d. L decomposes L_d into two sets, one consisting of those points which are closer to X than to Y and the other of those points which are closer to Y than to X. (Neither of these sets is empty. For since $m(X, Y)$ does not contain L_d, there is a point A in one of the sets. If we connect A to a point C of L, the points of $\underline{AC}$ on the other side of C from A will be in the other set). Therefore L must have dimension $d - 1$.[4] We shall see next:

(f) *For a given point P_0 and a given $d \leqq n$ one can always find s. l. $h_1, \cdots h_d$ through P_0 such that $\prod_{i=1}^{d} m(P_0, h_i)$ is an L_{n-d}.*

---

[3] See W. Hurewicz, "Sur la dimension des produits Cartésiens," *Annals of Mathematics*, vol. 36 (1935), pp. 194-197.

[4] For L is a linear space and therefore a μ-space. According to (d) it is homeomorphic to a Euclidean space.

Proof. This is obviously true for $n = 1, 2$. Assume it to be true for $n - 1$. Let h_d be any *s. l.* through P_0; $m(P_0, h_d)$ is an L_{n-1}. For any line h through P_0 in $m(P_0, h_d)$ the set $m(P_0, h_d) \cdot m(P_0, h)$ is the locus $\bar{m}(P_0, h)$ in this L_{n-1}. On account of the inductive assumption we can find lines $h_1, \cdots,$ h_{d-1} in $m(P_0, h_d)$ such that $\prod_{i=1}^{d} m(P_0, h_i)$ is an L_{n-d}.

We can now prove the linearity of the space:

(g) *Through $d + 1$, $d \leq n - 1$, points $P_0, \cdots, P_d$ of an n-dimensional μ-space, which are in no $L_{d'}$, with $d' < d$, there passes exactly one L_d.*

Proof. We consider the sets $m_i = m(P_0, \underline{P_0P_i})$. According to (e) m_2 either contains m_1 or intersects it in an L_{n-2}; we may say that m_2 intersects m_1 in an L_{n_2} with $n - 1 \geq n_2 \geq n - 2$. In the same way m_3 intersects L_{n_2} in an L_{n_3} with $n - 1 \geq n_3 \geq n - 3$ and so forth, finally m_d intersects $L_{n_{d-1}}$ in an L_{n_d} with $n - 1 \geq n_d \geq n - d$. All *s. l.* in this L_{n_d} through P_0 are perpendicular to all $\underline{P_0P_i}$. Now (f) shows that in L_{n_d} lines $h_1, \cdots, h_{n_d}$ through P_0 exist such that the sets $m(P_0, h_i)$ in this L_{n_d} intersect only at P_0, and as a consequence of (c) no $m(P_0, h_i)$ will contain the product of the others. It then follows again from (e) that the sets $m(P_0, h_i)$ intersect in an L_{n-n_d}, which contains all $\underline{P_0P_i}$. Since $P_0, \cdots, P_d$ are in no $L_{d'}$ with $d' < d$, we must have $n_d = n - d$ and for the same reason there cannot be two different L_d through these points.

4. In this section we shall establish the *Euclidean or Hyperbolic character of the metric.* For this purpose it will be sufficient to prove that a two-dimensional μ-space is either Euclidean or Hyperbolic. For any three points are in an L_2, according to the last theorem, and this L_2 is itself a μ-space (compare (e)) and herefrom one concludes easily that the whole space is either Euclidean or Hyperbolic. If the dimension of the space is $n \geq 3$, the Theorem of Desargues holds in every L_2 and the proof is trivial. But if the whole space is two-dimensional the Theorem of Desargues has to be proved (at least implicitly) and this accounts for the comparative lengthiness of the following considerations.

We call *motion of a metric space any single-valued mapping of the space onto itself which preserves distances.* Obviously such a mapping is topological.

Let now L be a two-dimensional μ-space and g an *s. l.* in it. To every point P not on g there exists exactly one point P' such that $g = m(P, P')$. For on account of (a), (b) the point P' can be determined as follows: Let F be the foot of P on g, then P' will be the point on $\underline{PF}$ with $PP' = 2PF = 2P'F$. We complete this correspondence by mapping every point of g onto

itself. We thus get an involutoric mapping R_g of L onto itself, which we call a *reflection in* g. The image σ' of a set σ under R_g will be designated by σR_g. Our aim is to show that *reflections are motions.* We fix a definite *s. l.* g, put generally $X' = XR_g$, and show that R_g is a motion. The proof will consist of several steps and we show first

(h) *If the line* $\underline{PQ}$ *intersects* g *then* $PQ = P'Q'$,

Proof. Let PQ intersect g at S. We have $PS = P'S$ and $QS = Q'S$ because $g = m(P, P') = m(Q, Q')$. Therefore (h) is true if S coincides with P or Q. If (PSQ) one has

$$P'Q' \leq P'S + SQ' = PS + SQ = PQ.$$

P and Q being on different sides of g, the points P' and Q' will also be on different sides hence $\underline{P'Q'}$ will contain a point S' of g, and one concludes in the same way that $PQ \leq P'Q'$. If (SPQ), then

$$P'Q' \geq SQ' - SP' = SQ - SP = PQ$$

But in this case we do not know yet that $\underline{P'Q'}$ also intersects g. The *s. l.* $\underline{PP'}$ intersects $\overline{SQ'}$ in a point R'. If $P' \subset \overline{R'P}$ then $\underline{P'Q'}$ obviously intersects g and we can conclude $PQ \geq P'Q'$ as above. Assume therefore that $P' \not\subset \overline{R'P}$. The image R of R' under R_g is on $\overline{PR'}$, hence $\underline{QR}$ intersects g in a point S_1. We conclude from what has already been proved that $QR = Q'R'$. We should then have

$$QS < QR + RS = Q'R' + R'S = Q'S.$$

An immediate consequence hereof is:

(k) *Under the reflection* R_g *an s. l.* h *which intersects* g *at* S *is mapped congruently onto an s. l.* h' *intersecting* g *at the same point* S.

We shall show next

(l) *If an s. l.* h *intersecting* g *at* S *is bisector of two points* P, Q $(h = m(P, Q))$ *then its image* $h' = hR_g$ *is bisector of the corresponding points* P', Q'; *or*

$$R_h R_g R_{h'} = R_g.$$

Proof. We have $P'S = PS = QS = Q'S$. If T is on h sufficiently close to S the *s. l.* $\underline{PT}$ and $\underline{QT}$ will intersect g in points L and M respectively. It follows from (k) that the images of $\underline{PL}$ and $\underline{QM}$ are $\underline{P'L}$ and $\underline{P'M}$ and that these *s. l.* intersect at the image T' of T on h'. Therefore $P'T' = PT = QT =$

$Q'T'$ and h' contains the two points S and T' of $m(P', Q')$, hence $h' = m(P', Q')$.

Call *s. l.* intersecting g *s. l.* of the first kind. Assume *s. l.* of the $(n-1)$-st kind have already been defined. Then we say an *s. l.* k is of the n-th kind if an *s. l.* h of the $(n-1)$-st kind intersecting k exists such that kR_h (which according to (k) is also an *s. l.*) is also of the $(n-1)$-st kind. An *s. l.* of the n-th kind is an *s. l.* of the m-th kind for all $m \geqq n$. For $\nu = 1$ the following statement is contained in (k) and (l).

(m) *Under R_g an s. l. k of the ν-th kind is mapped congruently onto an s. l. k' of the ν-th kind and one has*

$$R_k R_g R_{k'} = R_g.$$

We assume (m) to be true for $n-1$ and we shall prove it for n.

Since k is of the n-th kind a *s. l.* h of the $(n-1)$-st kind intersecting k at a point T exists such that $l = kR_h$ is also of the $(n-1)$-st kind. Therefore, by hypothesis, l and h go under R_g into *s. l.* l' and h' of the $(n-1)$-st kind which intersect at the image T' of T. $R_{h'}$ maps l' congruently onto a line k' (see (k)), and we have

$$k' = l'R_{h'} = lR_gR_{h'} = kR_hR_gR_{h'} = kR_g.$$

Each step means a congruent mapping of the *s. l.* in question, hence k is mapped congruently onto k', and k' is of the n-th kind because h' and $l' = h'R_h$ are of the $(n-1)$-st kind. Finally let P and Q be any two points for which $k = m(P, Q)$. In order to show that $R_kR_gR_{k'} = R_g$ we have to prove that $k' = m(P', Q')$ where $P' = PR_g$, $Q' = QR_g$. Now it follows from (l) that $l = kR_h = m(PR_h, QR_h)$ and from the inductive assumption that $R_lR_gR_{l'} = R_g$ or $lR_g = l' = m(PR_hR_g, QR_hR_g)$ and finally from (l) and (m) for $n-1$ that

$$k' = l'R_{h'} = m(PR_hR_gR_{h'}, QR_hR_gR_{h'}) = m(P', Q')$$

This completes the proof of (m). We shall see next:

(n) *Every s. l. is of some finite kind.*

To prove this we need as auxiliary result, that the *circles* of our metric *have finite length* in our metric. For this purpose it is sufficient to show that they are convex curves, since the proof that a convex curve has finite length, is identical with the known elementary proof for the same fact in Euclidean geometry.

Let e be any *s. l.*, Q a point not on e, F the foot of Q on e so that $e = m(F, \underline{QF})$. It follows herefrom that as X traverses one of the rays, into which F decomposes e, from F towards infinity, QX increases monotonically.

Therefore one has for any three points A, B, C with (A, B, C) on e the inequality $QB < \max\ (QA, QC)$, and since e is arbitrary, this holds for any three points with (A, B, C). Hence the circles with center Q are convex.

To prove (n) let g be the fixed line considered before, h_1 any *s. l.* not intersecting g. We want to prove that h_1 is of some finite kind. Let Q be any point on h_1. We draw the circle γ of radius l around Q, and two rays issuing from Q which intersect g, they may intersect γ in A and B. Let X traverse γ from A to B in such a way that all rays $\overrightarrow{QX}$ intersect g, and let then X continue on γ beyond B until it coincides for the first time with a point H_1 of h_1. Call γ_1 the arc of γ traversed by X from A to H_1. On γ_1 we choose a point H_2 so close to H_1 that $H_1H_2 < \frac{1}{2}AB$. Reflecting h_1 in $h_2 = \underline{QH_2}$ we get an *s. l.* $h_3 = \underline{QH_3}$, where H_3 is still on γ_1. We have $H_1H_2 = H_2H_3$ (see (h)) and $H_1H_2 + H_2H_3 = 2H_1H_2$ is smaller than the length of γ_1. If H_2 and H_3 are both between A and B we stop, otherwise we reflect h_2 in h_3 and get an *s. l.* $h_4 = \underline{QH_4}$ where H_4 is still on γ_1, $H_3H_4 = H_1H_2$ and $3H_1H_2$ is smaller than the length of γ_1. Since γ_1 has finite length we shall arrive at a first subscript ν such that H_ν and $H_{\nu+1}$ are both between A and B on γ_1. Putting $h_i = \underline{QH_i}$ we have

$$h_i R_{h_{i+1}} = h_{i+2}, \qquad i = 1, \cdots, \nu - 1.$$

$h_{\nu+1}$ and h_ν intersect g, hence $h_{\nu-1}$ is of the second kind, h_ν and $h_{\nu-1}$ being of the second kind, $h_{\nu-2}$ is of the third kind, and so forth. Finally, we see that h_1 is of the ν-th kind, q. e. d. (m) and (n) prove that reflections are motions.

Let now X and Z be any two points which have the same distance from a given point S. The points X and Z divide the circle with center S and radius $KS = ZS$ into two arcs, let γ_0 be one of them. From the convexity of γ_0 we conclude by continuity considerations that points H_1, Y_1, H_2 on γ_0 in this order can be found such that

$$XH_1 = H_1Y = YH_2 = H_2Z$$

Then

$$Y = XR_{\underline{SH_1}}, \qquad Z = YR_{\underline{SH_2}} \quad \text{hence } Z = XR_{\underline{SH_1}}R_{\underline{SH_2}}$$

$R_{\underline{SH_1}} R_{\underline{SH_2}}$ is a motion which preserves orientation and leaves S fixed, it is therefore a *rotation* which carries X into Z. Since, except for the condition $XS = ZS$ the points X, Z, S are arbitrary, we see that *our metric admits the full group of rotations around the arbitrary point S.* It is a very special case of a well known theorem by Hilbert [5] that under these conditions the metric is either Euclidean or Hyperbolic. Of course, one does not have to refer to

[5] "Ueber die Grundlagen der Geometrie," *Mathematische Annalen*, vol. 56 (1902), pp. 381-422.

this theorem; it is very simple to give a direct proof for the validity of the congruence axioms. This completes the proof of Theorem 1.

5. We are now going to discuss the application to motions mentioned in the introduction. We consider a finitely compact metric space in which for any two points exactly one *s. l.* through them exists. Assume that for any two congruent triples of points A, B, C and A', B', C', ($AB = A'B'$, $BC = B'C'$, $CA = C'A'$) a motion of the space exists under which A, B, C go into A', B', C' respectively.

Let h be any *s. l.*, a motion which leaves all points of h fixed will be called a rotation around h. Let A, B be any two points of h, and let C and C' be such that $AC = AC'$, $BC = BC'$ (or $A + B \subseteq m(C, C')$). We say the metric admits the full group of motions around h if for any four such points A, B, C, C' a rotation around h exists which carries C into C', or, which amounts to the same, if a motion of the space exists carrying A, B, C into A', B', C'. We can say:

If for any two congruent triples of points a motion exists carrying the first triple into the second, then the metric admits the full group of rotations around every straight line.

Furthermore one sees immediately that a space which admits the full group of rotations around every *s. l.* is a μ-space. For let $A \neq B$ and $A + B \subset m(C, C')$ then $AC = AC'$, $BC = BC'$, therefore a rotation around $\underline{AB}$ exists carrying C into C'. Every point X of $\underline{AB}$ remains fixed, therefore $XC = XC'$, hence $X \subset m(C, C')$, or $\underline{AB} \subset m(C, C')$. Using Theorem 1 we see that our space is the Euclidean or Hyperbolic space. So we have the

THEOREM 2. *If in a finitely compact metric space any two points can be connected by exactly one straight line and if the space admits the full group of rotations around every straight line, it is congruent to a finite dimensional Euclidean or Hyperbolic space. These rotations will always exist, if for any two congruent triples of points a motion of the space exists carrying the first triple into the second.*

The question arises whether it would be sufficient to assume that for any two congruent pairs of points a motion exists carrying the first pair into the second. For 2-dimensional space this is evidently sufficient; other results make it appear likely that it is also sufficient for 3-dimensional spaces; the question seems to be open for more than 3-dimensional spaces in spite of the existence of Riemann spaces of non-constant curvature in which a given line element can be carried into an arbitrary other one by a motion of the Riemann space.

THE JOHNS HOPKINS UNIVERSITY.

THE AXIS QUADRICS AT A POINT OF A SURFACE.*

By M. L. MacQueen.

1. Introduction. The purpose of this note is to define and study two quadrics, called axis quadrics, which are associated with each point of a given conjugate net on an analytic surface in ordinary projective space. In order to formulate a definition, let us consider a point x of a surface S referred to a conjugate net N_x. The osculating planes of the parametric curves at the point x intersect in the axis of the point x with respect to the net N_x. The osculating quadric along a generator at the point x of the ruled surface of axes constructed at the points of the u-curve through the point x is the limit of the quadric determined by the axis of the point x and the axes of two neighboring points P_1, P_2 on the u-curve as each of these points independently approaches the point x along the curve. The quadric thus defined will be called the axis quadric Q_u at the point x. A second axis quadric Q_v is defined similarly by using three consecutive axes of points on the v-curve through the point x. We shall now derive the equations of the axis quadrics and deduce some of their properties.

2. Equations. Let the surface S under consideration be an analytic non-ruled surface whose parametric vector equation, referred to conjugate parameters u, v, is

$$x = x(u, v). \tag{1}$$

The four coordinates x of a point on the surface and the four coordinates y of the point which is the harmonic conjugate of the point x with respect to the foci of the axis of the point x satisfy a completely integrable system of partial differential equations of the form [1]

$$\begin{aligned} x_{uu} &= px + \alpha x_u + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ x_{vv} &= qx + \delta x_v + Ny \qquad (LN \neq 0). \end{aligned} \tag{2}$$

The coefficients of these equations are functions of u, v and satisfy certain integrability conditions which need not be written here.

* Received June 21, 1940.

[1] E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, Chicago, 1932, p. 138.

It is easy to verify that

$$(3) \qquad y_u = fx - nx_u + sx_v + Ay, \qquad y_v = gx + tx_u + nx_v + By,$$

where we have placed

$$(4) \qquad \begin{aligned} fN &= c_v + ac + bq - c\delta - q_u, & gL &= c_u + bc + ap - c\alpha - p_v, \\ -nN &= a_v + a^2 - a\delta - q, & tL &= a_u + ab + c - \alpha_v, \\ sN &= b_v + ab + c - \delta_u, & nL &= b_u + b^2 - b\alpha - p, \\ A &= b - (\log N)_u, & B &= a - (\log L)_v. \end{aligned}$$

The ray-points, or Laplace transformed points, ρ, σ of the point x are given by the formulas

$$(5) \qquad \rho = x_u - bx, \qquad \sigma = x_v - ax.$$

Some of the invariants of the parametric conjugate net are given by

$$(6) \qquad \begin{aligned} 8\,\mathfrak{B}' &= 4a - 2\delta + (\log r)_v, & 8\,\mathfrak{C}' &= 4b - 2\alpha - (\log r)_u, \\ \mathfrak{H} &= sN, & \mathfrak{K} &= tL, \\ \mathfrak{D} &= -2nL, & r &= N/L, \\ P &= f + as - bn, & Q &= g + bt + an. \end{aligned}$$

We shall suppose that $\mathfrak{H}\mathfrak{K} \neq 0$, so that the parametric curves are not plane curves.

Any point X near the point x and on the u-curve through the point x may be defined by the following power series in the increment Δu:

$$(7) \qquad X = x + x_u \Delta u + \tfrac{1}{2} x_{uu} \Delta u^2 + \cdots.$$

If the points x, x_u, x_v, y are used as the vertices of a local tetrahedron of reference, with unit point suitably chosen, then any point given by an expression of the form

$$(8) \qquad x_1 x + x_2 x_u + x_3 x_v + x_4 y$$

has local coordinates proportional to $x_1, \cdots, x_4$. We find, by use of equations (2), that the local coordinates $x_1, \cdots, x_4$ of the point X are given by the expansions

$$(9) \qquad \begin{aligned} x_1 &= 1 + \tfrac{1}{2} p \Delta u^2 + \cdots, \\ x_2 &= \Delta u + \tfrac{1}{2} \alpha \Delta u^2 + \cdots, \\ x_3 &= \tfrac{1}{6} s L \Delta u^3 + \cdots, \\ x_4 &= \tfrac{1}{2} L \Delta u^2 + \cdots. \end{aligned}$$

The osculating plane of the u-curve at the point X is determined by the points X, X_u, X_{uu}, where

$$X_u = x_u + x_{uu}\Delta u + \tfrac{1}{2}x_{uuu}\Delta u^2 + \cdots,$$
$$X_{uu} = x_{uu} + x_{uuu}\Delta u + \tfrac{1}{2}x_{uuuu}\Delta u^2 + \cdots. \tag{10}$$

Similarly, the osculating plane of the v-curve at the point X is determined by the points X, X_v, X_{vv}, where

$$X_v = x_v + x_{uv}\Delta u + \tfrac{1}{2}x_{uuv}\Delta u^2 + \cdots,$$
$$X_{vv} = x_{vv} + x_{uvv}\Delta u + \tfrac{1}{2}x_{uuvv}\Delta u^2 + \cdots. \tag{11}$$

It is possible to express every derivative of x uniquely as a linear combination of x, x_u, x_v, y, so that the power series which represent the local coordinates of the points defined by equations (10), (11) are easily obtained and will not be written here. Making use of these results, we find that the local equations of the osculating planes of the u-curve and v-curve at the point X are respectively

$$\sum_{i=1}^{4} a_i x_i = 0, \qquad \sum_{i=1}^{4} b_i x_i = 0, \tag{12}$$

where the coefficients a_i, b_i are given by the expansions

$$\begin{aligned}
a_1 &= -\tfrac{1}{6}sL^2\Delta u^3 + \cdots,\\
a_2 &= \tfrac{1}{2}sL^2\Delta u^2 + \cdots,\\
a_3 &= L + L(\alpha + b - l_u)\Delta u + \cdots,\\
a_4 &= -sL\Delta u - \tfrac{1}{2}sL[2\alpha + 2b + (\log sL)_u - l_u]\Delta u^2 + \cdots.\\
b_1 &= N\Delta u + \cdots,\\
b_2 &= -N - 2bN\Delta u + \cdots,\\
b_3 &= aN\Delta u + \tfrac{1}{2}N[\alpha_v + 2ab - 2c + tL]\Delta u^2 + \cdots,\\
b_4 &= -nN\Delta u - \tfrac{1}{2}N[P + \alpha n + 4bn + n(\log nN)_u]\Delta u^2 + \cdots,
\end{aligned}$$

and l is defined by placing $l = \log r$.

The osculating planes (12) of the parametric curves at the point X intersect in the axis of the point X. It is easy to verify that the axis of the point X pierces the face $x_1 = 0$ of the tetrahedron of reference in the point Y whose local coordinates $y_1, \cdots, y_4$ are represented by the series

$$\begin{aligned}
y_1 &= 0,\\
y_2 &= -nLN\Delta u - \tfrac{1}{2}LN[f - as + 3\alpha n + 5bn - nl_u + n(\log nL)_u]\Delta u^2 + \cdots,\\
y_3 &= sLN\Delta u + \tfrac{1}{2}sLN[2\alpha + 6b + (\log sL)_u - l_u]\Delta u^2 + \cdots,\\
y_4 &= LN + LN(\alpha + 3b - l_u)\Delta u + \cdots,
\end{aligned} \tag{13}$$

Any point Z on the axis of the point X can be defined by a linear combination of the form

$$Z = hX + kY \qquad (h, k \text{ scalars}). \tag{14}$$

In order to calculate power series expansions for the local coordinates $z_1, \cdots, z_4$ of the point Z, it is sufficient to multiply the series (9) by h and the series (13) by k and add corresponding series. Demanding that the equation of a general quadric be satisfied by the power series thus calculated, identically in h, k and identically in Δu as far as the terms of the second degree, we obtain *the equation of the axis quadric Q_u referred to the tetrahedron x, x_u, x_v, y, namely,*

$$(15)\quad \begin{aligned}(f - as - bn + n_u + nN_u/N)x_3^2 - 2sx_1x_3 \\ + s(s_u/s + N_u/N - \alpha)x_2x_3 + 2s^2x_2x_4 + 2snx_3x_4 = 0.\end{aligned}$$

The coefficients in equation (15) are not all invariants because the tetrahedron of reference is not a covariant tetrahedron. For the purpose of writing the equation of the quadric Q_u referred to the covariant tetrahedron x, ρ, σ, y a simple computation shows that it is sufficient to replace x_1 in equation (15) by $x_1 - bx_2 - ax_3$. If we make this substitution and simplify the coefficients by means of equations (6), we arrive at *the equation of the axis quadric Q_u referred to the covariant tetrahedron x, ρ, σ, y, namely,*

$$(16)\quad [NP - \tfrac{1}{2}(r\mathfrak{D})_u]x_3^2 - 2\mathfrak{H}x_1x_3 + \mathfrak{H}Ix_2x_3 + 2s\mathfrak{H}x_2x_4 + 2n\mathfrak{H}x_3x_4 = 0,$$

where I is defined by placing

$$(17)\quad I = (\log \mathfrak{H})_u + 4\mathfrak{C}' + \tfrac{1}{2}l_u.$$

The equation of the quadric Q_v at the point x can be written immediately by interchanging u and v and making the necessary symmetrical interchanges of the other symbols. For this result we find

$$(18)\quad [LQ + \tfrac{1}{2}\mathfrak{D}_v]x_2^2 - 2\mathfrak{K}x_1x_2 + \mathfrak{K}Jx_2x_3 + 2t\mathfrak{K}x_3x_4 - 2n\mathfrak{K}x_2x_4 = 0,$$

where J is defined by

$$(19)\quad J = (\log \mathfrak{K})_v + 4\mathfrak{B}' - \tfrac{1}{2}l_v.$$

3. Properties. Some simple properties of the quadrics Q_u and Q_v will now be deduced. In the first place, it is clear that the quadric Q_u intersects the tangent plane, $x_4 = 0$, in the u-tangent, $x_3 = x_4 = 0$, and in the line represented by the equations

$$(20)\quad 2\mathfrak{H}x_1 - \mathfrak{H}Ix_2 - [NP - \tfrac{1}{2}(r\mathfrak{D})_u]x_3 = 0, \qquad x_4 = 0.$$

Likewise, the quadric Q_v intersects the tangent plane in the v-tangent, $x_2 = x_4 = 0$, and in the line

$$(21) \qquad 2\mathfrak{K}x_1 - [LQ + \tfrac{1}{2}\mathfrak{D}_v]x_2 - \mathfrak{K}Jx_3 = 0, \qquad x_4 = 0.$$

The line (20) passes through the ray-point ρ in case $I = 0$, and the line (21) passes through the ray-point σ in case $J = 0$.

The intersections of the quadrics Q_u, Q_v with the osculating planes of the parametric curves at the point x are of some interest. It is evident that the axis, $x_2 = x_3 = 0$, of the point x is a common generator of the two quadrics. The osculating plane, $x_3 = 0$, of the u-curve is tangent to the quadric Q_u at the point x. Three vertices x, ρ, and y of the tetrahedron of reference x, ρ, σ, y lie on the quadric Q_u. The fourth vertex σ also lies on this quadric if, and only if, $NP - \frac{1}{2}(r\mathfrak{D})_u = 0$. Moreover, the osculating plane, $x_2 = 0$, of the v-curve at the point x contains two generators of the quadric Q_u, namely, the axis and the generator whose equations are

$$(22) \qquad 2\mathfrak{H}x_1 - [NP - \tfrac{1}{2}(r\mathfrak{D})_u]x_3 - 2n\mathfrak{H}x_4 = 0, \qquad x_2 = 0.$$

This line meets the line (20) in the point

$$(23) \qquad (NP - \tfrac{1}{2}(r\mathfrak{D})_u, 0, 2\mathfrak{H}, 0),$$

and intersects the axis in the point whose coordinates are

$$(24) \qquad (n, 0, 0, 1).$$

Similarly, in the osculating plane $x_3 = 0$, we easily find that the generator of the quadric Q_v that corresponds to (22) intersects the line (21) in the point

$$(25) \qquad (LQ + \tfrac{1}{2}\mathfrak{D}_v, 2\mathfrak{K}, 0, 0),$$

and meets the axis in the point

$$(26) \qquad (-n, 0, 0, 1).$$

The points x and y are separated harmonically by the points (24), (26). Furthermore, the osculating plane of the v-curve at the point x is tangent to the quadric Q_u at the point (24), and the osculating plane of the u-curve is tangent to the quadric Q_v at the point (26).

The quadrics Q_u and Q_v intersect, besides in the axis, also in a twisted cubic which has the axis for a bisecant. This cubic meets the tangent plane in points (23), (25), and in the point of intersection of the lines (20), (21), since these points are common to both quadrics. Eliminating x_1 between equations (16), (18), we obtain the equation of the cubic cone projecting from the point x the curve of intersection of the quadrics Q_u, Q_v, namely,

$$(27)\quad \mathfrak{R}[NP - \tfrac{1}{2}(r\mathfrak{D})_u - \mathfrak{H}J]x_2x_3^2 - \mathfrak{H}[LQ + \tfrac{1}{2}\mathfrak{D}_v - \mathfrak{R}I]x_2^2x_3 + 2\mathfrak{H}\mathfrak{R}(sx_2^2 + 2nx_2x_3 - tx_3^2)x_4 = 0.$$

The form of this equation makes it evident that the axis of the point x is a double line of this cone. Moreover, the equation of the nodal tangent planes of the cone along the axis is obtained by setting equal to zero the coefficient of x_4 in equation (27). Let us recall that the curvilinear differential equation of the axis curves of the net N_x is given by

$$s\,du^2 + 2n\,du dv - t\,dv^2 = 0.$$

Thus the following theorem is proved:

The nodal tangent planes of the cone (27) along the axis of the point x are the planes through the axis that cut the tangent plane in the tangents of the axis curves.

Furthermore, the cone (27) cuts the tangent plane in the parametric tangents at the point x and in the line

$$\mathfrak{H}[LP + \tfrac{1}{2}\mathfrak{D}_v - \mathfrak{R}I]x_2 - \mathfrak{R}[NP - \tfrac{1}{2}(r\mathfrak{D})_u - \mathfrak{H}J]x_3 = 0, \qquad x_4 = 0,$$

which joins the point x to the point of intersection of the lines defined by equations (20), (21).

Now let us project the curve of intersection of the quadrics Q_u, Q_v from the ray-point σ. The result of eliminating x_3 between equations (16), (18) is found to be a composite quartic cone, one component being the osculating plane, $x_2 = 0$, of the v-curve at the point x. This projecting cone meets the osculating plane, $x_3 = 0$, of the u-curve in the axis and in a plane cubic curve which intersects the axis in the point x and in the points

$$(28)\qquad y \pm (n^2 + st)^{\frac{1}{2}}x.$$

Since the points defined by the formulas (28) are the foci of the axis, we arrive at the following conclusion:

The quadrics Q_u, Q_v intersect in the axis of the point x and in a cubic curve which intersects the axis in the foci of the axis.

In his investigation of the osculating linear complexes of the two curves

of a conjugate net through a point of a surface, Lane has observed [2] that the point ρ corresponds to the plane

$$NIx_3 + 2\mathfrak{H}x_4 = 0 \tag{29}$$

in the null system of the osculating linear complex of the u-curve. The plane (29) is found to be tangent to the quadric Q_u at the point ρ. Therefore *the tangent plane of the quadric Q_u at the ray-point ρ is the plane which corresponds to the point ρ in the null system of the osculating linear complex of the u-curve.*

Finally, the polar plane of the ray-point σ with respect to the quadric Q_u has the equation

$$\mathfrak{H}x_1 - \tfrac{1}{2}\mathfrak{H}Ix_2 - [NP - \tfrac{1}{2}(r\mathfrak{D})_u]x_3 - n\mathfrak{H}x_4 = 0. \tag{30}$$

This plane passes through the point y if, and only if, $n = 0$, so that the parametric net is harmonic.

SOUTHWESTERN COLLEGE.

---

[2] E. P. Lane, "Contributions to the theory of conjugate nets," *American Journal of Mathematics*, vol. 49 (1927), p. 575.

A CRITERION FOR SOLVABILITY BY RADICALS.*[1]

By B. W. Brewer.

Introduction. The Galois criterion for solvability by radicals is valid in the fields of characteristic zero, but not in those of prime characteristic. There is given in this paper a criterion which is valid in any field. This criterion emphasizes the importance of the primitive roots of unity and the cyclotomic polynomial in the theory of solvability by radicals.

1. The cyclotomic polynomial in a field of prime characteristic. We establish in this section certain properties of the cyclotomic polynomial in a field of prime characteristic which are essential to the development.

An absolutely algebraic field of prime characteristic is uniquely defined by its characteristic and absolute degree.[2] We shall denote by $A_{p,m}$ the absolutely algebraic field of prime characteristic p and absolute degree m. However, we shall denote by $GF[p^m]$ the finite field of p^m elements when it seems necessary to emphasize its finite character.

Lemma. *An irreducible polynomial $f(x)$ of degree n in the $A_{p,m}$ factors in the $A_{p,m'}$, m a divisor of m', into δ distinct irreducible factors each of degree n/δ, δ being the greatest common divisor of n and m'/m.*

Proof. The coefficients of $f(x)$ are elements of some $GF[p^\kappa] \subseteqq A_{p,m}$. Since $f(x)$ is irreducible in the $A_{p,m}$, $f(x)$ is irreducible in the $GF[p^\kappa]$. From the fact that κ is a divisor of m, and δ a divisor of m'/m, it follows that $GF[p^{\kappa\delta}] \subseteqq A_{p,m'}$. Since the lemma is known to be true for finite fields,[3] $f(x)$ factors in the $GF[p^{\delta\kappa}]$ into δ distinct irreducible factors $\phi_1(x), \phi_2(x), \cdots, \phi_\delta(x)$ each of degree n/δ. These are the irreducible factors of $f(x)$ in the $A_{p,m'}$. To show this, let $f(x) = \psi_1(x)\psi_2(x) \cdots \psi_s(x)$, where $\psi_1(x), \psi_2(x), \cdots, \psi_s(x)$ are irreducible in the $A_{p,m'}$. The coefficients of $\psi_1(x), \cdots, \psi(x)$ are elements of some $GF[p^r] \subseteqq A_{p,m'}$. Let $r = ab$, where a is a divisor of m, and b a divisor m'/m. Let $\nu_1\kappa$ be the least common multiple of a and κ, and $\nu_2\delta$ the least common multiple of b and δ. Since $\nu_1\kappa$ is a divisor of m, and $\nu_2\delta$ a divisor of m'/m, $GF[p^r] \subseteqq GF[p^{\nu\delta\kappa}] \subseteqq A_{p,m'}$, where $\nu = \nu_1\nu_2$. ν_1 is rela-

* Received March 7, 1939; revised September 1, 1940.

[1] Most of the results obtained in this paper were included in a dissertation for the doctorate, University of Missouri (1938).

[2] The absolute degree is a certain G-adic number. See E. Steinitz, *Algebraische Theorie der Körper*, Berlin, Walter de Gruyter & Co., 1930, pp. 79-88.

[3] L. E. Dickson, *Linear Groups*, Leipzig, B. G. Teubner, 1901, p. 33.

tively prime to n, since $f(x)$ is irreducible in the $GF[p^{\nu_1\kappa}]$, a subfield of the $A_{p,m}$. ν_2 is relatively prime to n/δ, since ν_2 is a divisor of m'/m, and n/δ is relatively prime to m'/m. Hence ν is relatively prime to n/δ. Thus, since $f(x)$ factors in the $GF[p^{\delta\kappa}]$ into the δ distinct irreducible factors $\phi_1(x), \cdots \phi_\delta(x)$, $f(x)$ factors in the $GF[p^{\nu\delta\kappa}]$ into these same irreducible factors. But the irreducible factors of $f(x)$ in the $GF[p^{\nu\delta\kappa}]$ are its irreducible factors in the $A_{p,m'}$. Hence $f(x)$ factors in the $A_{p,m'}$ into δ distinct irreducible factors each of degree n/δ, and the lemma is proved.

The set of all absolutely algebraic elements of a field K of prime characteristic constitutes a field which is absolutely algebraic. We shall call this field the *maximal absolutely algebraic subfield* of K.

Throughout this paper, $g_n(x)$ will denote, for a given positive integer n and field K, that cyclotomic polynomial in K, whose roots are the $\phi(n)$ distinct primitive n-th roots of unity, and T_n will denote the root field of $g_n(x)$ over K. If K is of prime characteristic p, this implies that $n \not\equiv 0 \pmod{p}$, since if $n \equiv 0 \pmod{p}$, primitive n-th roots of unity over K do not exist.

THEOREM 1. *Let K be a field of prime characteristic p, and m be the absolute degree of the maximal absolutely algebraic subfield of K. Let $g_n(x)$ be the cyclotomic polynomial in K, whose roots are the $\phi(n)$ distinct primitive n-th roots of unity, $n \not\equiv 0 \pmod{p}$. Then, if d is the greatest common divisor of $\phi(n)$ and m, and e is the exponent to which p^d belongs modulo n, $g_n(x)$ factors in K into $\phi(n)/e$ distinct, irreducible, separable factors each of degree e. Moreover, the Galois group of $g_n(x)$ relative to K is cyclic of order e.*

Proof. Since d is a divisor of m, the maximal absolutely algebraic subfield of K, namely the $A_{p,m}$, contains the $GF[p^d]$ as a subfield. Since theorem 1 is known to be true for finite fields,[4] $g_n(x)$ factors in the $GF[p^d]$ into $\phi(n)/e$ distinct, irreducible, separable factors each of degree e. But e is relatively prime to m/d, and hence it follows from the above lemma that these are the irreducible factors of $g_n(x)$ in the $A_{p,m}$. Moreover, these are the irreducible factors of $g_n(x)$ in K, since the coefficients of the irreducible factors of $g_n(x)$ in K are symmetric functions of certain of the primitive n-th roots of unity, and hence elements of the $A_{p,m}$. Since the Galois group H of $g_n(x)$ relative to the $GF[p^d]$ is cyclic of order e, and the common degree of the irreducible factors of $g_n(x)$ in K is e, it follows from the well-known

[4] H. Rauter, "Höhere Kreiskörper," *Journal für die reine und angewandte Mathematik*, vol. 159 (1928), pp. 220-227.

properties of the cyclotomic polynomial that H is the Galois group of $g_n(x)$ relative to K. Hence theorem 1 is proved.

We have immediately from theorem 1,

THEOREM 2. *Let K be a field of prime characteristic p, and m be the absolute degree of the maximal absolutely algebraic subfield of K. Let $g_n(x)$ be the cyclotomic polynomial in K, whose roots are the $\phi(n)$ distinct primitive n-th roots of unity, $n \not\equiv 0 \pmod p$. Then a necessary and sufficient condition that $g_n(x)$ be irreducible in K is that p be a primitive root of n, and $\phi(n)$ be relatively prime to m.*[5]

THEOREM 3. *Let K be a field of prime characteristic p, and m be a composite positive integer not divisible by p. Then $K \subseteqq T_d \subseteqq T_n$, where d is a divisor of n. Moreover, if d is equal to the product of the distinct prime factors $q_1, q_2, \cdots, q_r$ of n, the degree of T_n over T_d is a divisor of n.*

Proof. That $K \subseteqq T_d \subseteqq T_n$ is well known, and hence we have only to show that the degree of T_n over T_d (denoted by $[T_n : T_d]$) is a divisor of n if $d = q_1 q_2 \cdots q_r$.

Let m be the absolute degree of the maximal absolutely algebraic subfield of K. Since $\phi(d)$ is a divisor of $\phi(n)$, it follows from theorem 1 that $[T_d : K]$ is the exponent e to which $\bar{p} = p^{(\phi(n),m)}$ belongs modulo d. Hence $\bar{p}^e \equiv 1 \pmod d$. Now if a and b are relatively prime positive integers, and $a \equiv 1 \pmod b$, then $a^{b^{s-1}} \equiv 1 \pmod{b^s}$ $(s = 1, 2, \cdots)$. Hence it follows that $\bar{p}^{e(n/d)} \equiv 1 \pmod n$. Therefore, if $n = q_1^{k_1} q_2^{k_2} \cdots q_r^{k_r}$, the exponent to which $\bar{p}$ belongs modulo n is $e q_1^{s_1} q_2^{s_2} \cdots q_r^{s_r}$, where $0 \leqq s_i < k_i$ $(i = 1, 2, \cdots, r)$. Thus from theorem 1, $[T_n : K] = e q_1^{s_1} q_2^{s_2} \cdots q_r^{s_r}$ and therefore $[T_n : T_d] = q_1^{s_1} q_2^{s_2} \cdots q_r^{s_r}$, a divisor of n.

2. A criterion for solvability by radicals. An extension $\bar{K}$ of the field K is said to be *pure* over K, if and only if $\bar{K} = K(\alpha)$, α being a root of an irreducible binomial in K. We then have the

Definition. A polynomial $f(x)$ in a field K_0 is said to be *solvable by radicals* over K_0 if and only if there exists a chain of fields

$$K_0 \subset K_1 \subset \cdots \subset K_s, \quad K_s \supseteqq W_f,$$

where K_i is pure and of prime degree over K_{i-1} $(i = 1, 2, \cdots, s)$, and W_f is the root field of $f(x)$ over K_0.

[5] Compare with F. Levi, "Zur Reduzibilität der Kreisteilungspolynome," *Compositio Mathematica*, vol. 2 (1935), pp. 303-304.

The fact that primitive n-th roots of unity exist and that $g_n(x)$ is solvable by radicals over a field of characteristic zero, for every positive integer n, is made use of in the proof of the Galois criterion. But primitive n-th roots of unity do not exist over a field K of prime characteristic p if $n \equiv 0 \pmod{p}$, and if $n \not\equiv 0, \pmod{p}$, $g_n(x)$ may not be solvable by radicals over K. The recognition of these facts leads to the following criterion for solvability by radicals.

THEOREM 4. *Let $f(x)$ be a polynomial in a field K_0, and n be the order of the Galois group of $f(x)$ relative to K_0. Then $f(x)$ is solvable by radicals over K_0 if and only if*

(I) *G is solvable,*

(II) *primitive n-th roots of unity exist over K_0, and the cyclotomic polynomial $g_n(x)$ in K_0, whose roots are the $\phi(n)$ distinct primitive n-th roots of unity, is solvable by radicals over K_0.*

Proof. We first make several remarks concerning notation.

If N is a separable normal extension of finite degree over K_0, and $\bar{K}$ is any extension of K_0, then the root fields over $\bar{K}$ of all those polynomials in K_0 which have N as their common root field over K are one and the same separable normal extension $\bar{N}$ of finite degree over $\bar{K}$. This field $\bar{N}$ uniquely determined by K_0, N, and $\bar{K}$ will be denoted by $\{N, \bar{K}\}$. Now $N \subseteqq \{N, \bar{K}\}$, $\bar{K} \subseteqq \{N, \bar{K}\}$, and we shall denote the intersection of N and $\bar{K}$ by $[N, \bar{K}]$.

W_f will denote the root field of $f(x)$ over K_0, and M_f the maximal separable (necessarily normal) extension of K_0 contained in W_f. Then G is by definition the Galois group of W_f relative to K_0, and G is isomorphic to the Galois group of M_f relative to K_0.[6] This implies that the degree of M_f over K_0 is n.

T_m will denote, for a given positive integer m, the root field of $g_m(x)$ over K_0.

Now suppose (I) and (II) hold. We show that $f(x)$ is solvable by radicals over K_0.

Since (II) holds, primitive n-th roots of unity exist over K_0, and there exists a chain of fields

$$K_0 \subset K_1 \subset \cdots \subset K_r, \qquad K_r \supseteqq T_n,$$

[6] See B. L. van der Waerden, *Moderne Algebra*, vol. 1, sec. ed., Berlin, Julius Springer, 1937, pp. 125-129.

where K_i is pure and of prime degree over K_{i-1} $(i=1,2,\cdots,r)$ (If $T_n=K_0$, $r=0$). Since (I) holds, H is solvable, and hence there exists a chain of fields

$$K_r \subset K_{r+1} \subset \cdots \subset K_{r+s}=\{M_f, K_r\}, \qquad K_{r+s} \supseteqq M_f,$$

where K_{r+i} is normal and of prime degree q_i over K_{r+i-1} $(i=1,2,\cdots,s)$ (If $M_f \subseteqq K_r$, $s=0$). Since $K_r \supseteqq T_n$, and $n\equiv 0 \pmod{q_i}$ $(i=1,2,\cdots,s)$, it follows that $K_r \supseteqq T_{q_i}$ $(i=1,2,\cdots,s)$, and hence K_{r+i} is pure over K_{r+i-1} $(i=1,2,\cdots,s)$. If $M_f=W_f$, then $f(x)$ is solvable by radicals over K_0. If $M_f\neq W_f$, then K_0 is of prime characteristic p, and there exists a chain of fields.

$$M_f=\bar{K}_0 \subset \bar{K}_1 \subset \cdots \subset \bar{K}_v=W_f,$$

where $\bar{K}_i=\bar{K}_{i-1}(\alpha)$, α_i being a root of an irreducible binomial x^p-a_i, in $\bar{K}_{i-1}$ $(i=1,2,\cdots,v)$. Let $\tilde{K}_0=K_{r+s}$, $\tilde{K}_1=\tilde{K}_0(\alpha_1)$, and in general $\tilde{K}_i=\tilde{K}_{i-1}(\alpha_i)$ $(i=1,2,\cdots,v)$. Then either $\tilde{K}_i=\tilde{K}_{i-1}$ or $\tilde{K}_i$ is pure and of prime degree p over $\tilde{K}_{i-1}$ $(i=1,2,\cdots,v)$. Therefore there exists a chain of fields

$$\begin{aligned} K_0 \subset K_1 \subset \cdots \subset K_r \subset K_{r+1} \subset \cdots \subset K_{r+s} \\ \subset K_{r+s+1} \subset \cdots \subset K_{r+s+t}, \; K_{r+s+t} \supseteqq W_f, \end{aligned}$$

where K_i is pure and of prime degree over K_{i-1} $(i=1,2,\cdots,r+s+t)$. Hence in this case $f(x)$ is solvable by radicals over K_0.

Next suppose that $f(x)$ is solvable by radicals over K_0. We show that (I) and (II) hold.

If $n=1$, it is evident that (I) and (II) hold. Hence we shall suppose that $n\neq 1$, and let $p_1, p_2,\cdots,p_r$ be the distinct prime factors of n.

By our assumption, there exists a chain of fields

$$K_0 \subset K_1 \subset \cdots \subset K_s, \qquad K_s \supseteqq W_f, \tag{1}$$

where $K_i=K_{i-1}(\beta_i)$, β_i being a root of an irreducible binomial $x^{q_i}-b_i$ of prime degree q_i in K_{i-1} $(i=1,2,\cdots s)$. Let $q_{i_1}, q_{i_2},\cdots q_{i_e}$ be those primes found among $q_1, q_2,\cdots,q_s$ which are not equal to the characteristic of K_0. Then if $m=q_{i_1}q_{i_2}\cdots q_{i_e}$, primitive m-th roots of unity exist over K_0. As is well known, T_m is metacyclic over K_0, and hence there exists a chain of fields

$$K_0=\bar{K}_0 \subset \bar{K}_1 \subset \cdots \subset \bar{K}_t=T_m,$$

where $\bar{K}_i$ is normal and of prime degree over $\bar{K}_{i-1}$ $(i=1,2,\cdots,t)$. Let $\hat{K}_0=\bar{K}_t$, $\hat{K}_1=\hat{K}_0(\beta_1)$, and in general $\hat{K}_i=\hat{K}_{i-1}(\beta_i)$ $(i=1,2,\cdots,s)$. Then since $T_{q_{i_j}} \subseteqq \bar{K}_t$ $(j=1,2,\cdots,e)$, it follows from (1) that either

$\hat{K}_i = \hat{K}_{i-1}$ or $\hat{K}_i$ is pure and of prime degree over $\hat{K}_{i-1}$ $(i = 1, 2, \cdots, s)$. Hence there exists a chain of fields

$$K_0 = \bar{K}_0 \subset \bar{K}_1 \subset \cdots \subset \bar{K}_t \subset \bar{K}_{t+1} \subset \cdots \subset \bar{K}_{t+u}, \qquad \bar{K}_{t+u} \supseteqq M_f,$$

where $\bar{K}_i$ is normal and of prime degree over $\bar{K}_{i-1}$ $(i = 1, 2, \cdots, t + u)$. It follows that there exists a chain of fields

$$K_0 = \tilde{K} \subset \tilde{K}_1 \subset \cdots \subset \tilde{K}_v, \qquad \tilde{K}_v = M_f,$$

where $\tilde{K}_i$ is normal and of prime degree over $\tilde{K}_{i-1}$ $(i = 1, 2, \cdots, v)$.[7] Hence H is solvable, and likewise G. Thus (I) holds.

If K_0 is of characteristic zero, it is well known that (II) necessarily holds. Hence suppose that K_0 is of prime characteristic p. Since from (1), $K_s \supseteqq W_f$, there exists for each p_i $(i = 1, 2, \cdots, r)$ a $q_{j_i} = p_i$, such that $[\{M_f, K_{j_i-1}\}, K_{j_i}] = K_{j_i}$. Moreover, since M_f is separable over K_0, $[\{M_f, K_{j_i-1}\}, K_{j_i}]$ is separable over K_{j_i-1}, and being pure over K_{j_i-1}, cannot be of degree p over K_{j_i-1}. Hence $p_i \neq p$ $(i = 1, 2, \cdots, r)$, and thus primitive n-th roots of unity exist over K_0. Since $K_{j_i} = K_{j_i-1}(\beta_{j_i}) \subseteqq \{M_f, K_{j_i-1}\}$, and $\{M_f, K_{j_i-1}\}$ is normal over K_{j_i-1}, $x^{p_i} - b_{j_i}$ has all of its roots in $\{M_f, K_{j_i-1}\}$, a subfield of K_s $(i = 1, 2, \cdots, r)$. This implies that $T_{p_i} \subseteqq K_s$ $(i = 1, 2, \cdots, r)$, and hence that $T_d \subseteqq K_s$, where $d = p_1 p_2 \cdots p_r$. If $\{T_n, K_s\} = K_s$, then $g_n(x)$ is solvable by radicals over K_0, and the proof is complete. Hence suppose that $\{T_n, K_s\} \neq K_s$. Then it follows from theorem 3 that $[\{T_n, K_s\} : K_s]$ is a divisor of n. Thus, since $\{T_n, K_s\}$ is cyclic over K_s, and $T_{p_i} \subseteqq K_s$ $(i = 1, 2, \cdots, r)$, there exists a chain of fields

$$K_s \subset K_{s+1} \subset \cdots \subset K_{s+t} = \{T_n, K_s\},$$

where K_{s+i} is pure and of prime degree over K_{s+i-1} $(i = 1, 2, \cdots, t)$. But $T_n \subseteqq \{T_n, K_s\}$, and hence $g_n(x)$ is solvable by radicals over K_0, and (II) holds. This completes the proof of theorem 4.

If K_0 is of characteristic zero, theorem 4 reduces to the Galois criterion, and this criterion is equivalent to a number-theoretic condition on the index series of G.

If K_0 is of prime characteristic, theorem 4 is equivalent to a similar number theoretic condition on the index series of G. To show this, it is only necessary to prove the following two theorems concerning the cyclotomic polynomial.

---

[7] See H. Hasse. *Höhere Algebra* II, sec. ed., Walter de Gruyter & Co., 1937, theorem 119, p. 120.

THEOREM 5. *A necessary and sufficient condition that $g_n(x)$ be solvable by radicals over a field K_0 of prime characteristic p, n being composite and not divisible by p, is that $g_d(x)$ be solvable by radicals over K_0 for every prime divisor d of n.*

Proof. The necessity of the condition follows at once from the definition of solvability by radicals and theorem 3.

To show the sufficiency of the condition, let $p_1, p_2, \cdots, p_r$ be the distinct prime factors of n, and suppose that $g_{p_i}(x)$ is solvable by radicals over K_0 $(i = 1, 2, \cdots, r)$. Then it follows that there exists a chain of fields

$$K_0 \subset K_1 \subset \cdots \subset K_s, K_s \supseteqq T_{p_i} \qquad (i = 1, 2, \cdots, r)$$

where K_j is pure and of prime degree over K_{j-1} $(j = 1, 2, \cdots, s)$, and T_{p_i} is the root field of $g_{p_i}(x)$ over K_0 $(i = 1, 2, \cdots, r)$. As in the proof of theorem 4, this implies that $g_n(x)$ is solvable by radicals over K_0, and hence the condition is sufficient.

Let K be a field of prime characteristic p, and m be the absolute degree of the maximal absolutely algebraic subfield of K. Then by means of p and m, we can define a class $C_{p,m}$ of primes in the following recursive manner.

Let $p_1, p_2, \cdots$ be the set of all primes (including 1), where $p_i < p_j$ if $i < j$, and suppose that $p = p_k$.

1. For $i < k$, p_i belongs to $C_{p,m}$.

2. p does not belong to $C_{p,m}$.

3. For $i > k$, let e_i be the exponent to which $p^{(\phi(p_i), m)}$ belongs modulo p_i, and $p_{i_1}, p_{i_2}, \cdots, p_{i_s}$ be the distinct prime factors of e_i. Then $i_j < i$ $(j = 1, 2, \cdots, s)$, and if p_{i_j} belongs to $C_{p,m}$ $(j = 1, 2, \cdots, s)$, p_i belongs to $C_{p,m}$, otherwise p_i does not belong to $C_{p,m}$.

Then from theorems 1, 4, and 5, we have

THEOREM 6. *Let K be a field of prime characteristic p, and m be the absolute degree of the maximal absolutely algebraic subfield of K. Then a necessary and sufficient condition that $g_d(x)$ be solvable by radicals over K, d being a prime distinct from p, is that d belong to the class $C_{p,m}$.*

It is now evident from theorems 5 and 6 that theorem 4 is equivalent to the following theorem if K_0 is of prime characteristic.

THEOREM 7. *Let K be a field of prime characteristic p, and m be the absolute degree of the maximal absolutely algebraic subfield of K. Then a*

necessary and sufficient condition that a polynomial $f(x)$ in K be solvable by radicals over K is that the index series of the Galois group of $f(x)$ relative to K consist of prime numbers belonging to the class $C_{p,m}$.

We may now give a characterization of those fields of prime characteristic over which every cyclotomic polynomial is solvable by radicals.

THEOREM 8. *Let K be a field of prime characteristic p, and m be the absolute degree of the maximal absolutely algebraic subfield of K. Then a necessary and sufficient condition that the cyclotomic polynomial $g_n(x)$ in K, whose roots are the $\phi(n)$ distinct primitive n-th roots of unity, be solvable by radicals over K, for every $n \not\equiv 0 \pmod{p}$, is that m be divisible by p^∞.*

Proof. To show the necessity of the condition, suppose that $g_n(x)$ is solvable by radicals over K for every $n \not\equiv 0 \pmod{p}$. Moreover, suppose that the exponent d of p in m is finite. We show that this last assumption leads to a contradiction. If $k = p^{p^{d+1}} - 1$, then $k \not\equiv 0 \pmod{p}$, and the exponent to which $p^{(\phi(k),m)}$ belongs modulo k is p. Hence from theorems 1 and 7, $g_k(x)$ is not solvable by radicals over K. Thus we have a contradiction, and the condition is necessary.

To show the sufficiency of the condition, we have only to note, in view of theorems 5 and 6, that if m is divisible by p^∞, every prime distinct from p belongs to the class $C_{p,m}$.

UNIVERSITY OF MISSOURI.

ACCESSIBILITY AND SEPARATION BY SIMPLE CLOSED CURVES.*[1]

By Ebon E. Betz.

The accessibility theorem presented here was obtained in an attempt to determine whether or not a compact Peano space[2] M need be a simple closed surface if M is separated by each of its simple closed curves but by no pair of its points. From this accessibility theorem it follows immediately that in such a space M, if the number of complementary domains of each simple closed curve is finite, then every point of the boundary of a complementary domain of a simple closed curve is regularly accessible[3] from that domain. It follows readily that the space M described above is a simple closed surface if the complement of every simple closed curve of M consists of exactly two components. This result was obtained previously by Zippin by a different method.[4] Using this accessibility theorem, Dr. D. W. Hall has extended Zippin's result, proving that the above space M is a simple closed surface if there exists a number N such that no simple closed curve of M has more than N complementary domains.

I am deeply indebted to Professor J. R. Kline for his guidance and aid

---

* Received June 10, 1940.

[1] Presented to the American Mathematical Society, April, 1939.

[2] A Peano space is a non-degenerate, connected, locally connected, locally compact, metric space.

[3] A boundary point P of a point set R is said to be *accessible* from R if there exists an arc having P as one endpoint and contained in R except for the point P. P is *regularly accessible* from R if for every positive number e there exists a positive number d such that every point of $R \cdot S(P, d)$ can be joined to P by an arc of $S(P, e)$ lying in R except for P.

Some of the symbols used are listed here for reference.

$\rho(X, Y)$ denotes the distance from X to Y, whether X and Y are points or point sets.

$S(P, e)$ denotes the set of points x such that $\rho(P, x) < e$.

$F(P, e)$ denotes the set of points x such that $\rho(P, x) = e$.

$\bar{S}(P, e)$ denotes the closure of $S(P, e)$.

$\supset$ means "contains"; $\subset$ means "is contained in."

$F(D)$ denotes the set of boundary points of the point set D; that is, $F(D) = \bar{D} \cdot \overline{(M - D)}$, where M denotes the space of which D is a subset.

If t is an arc, then $\langle t \rangle$ denotes the same arc with its endpoints deleted. If ab is an arc from a to b, then $\langle ab$ and $ab\rangle$ denote $ab - a$ and $ab - b$ respectively.

[4] Leo Zippin, "On continuous curves and the Jordan curve theorem," *American Journal of Mathematics*, vol. 52 (1930), pp. 331-350. See Theorem 4, p. 348.

in the work on this problem, which he suggested. Dr. Dick Wick Hall has also aided greatly.

LEMMA 1. *If the limit point P of the arcwise connected set R is not regularly accessible from R, then there exists a positive number ϵ such that to every pair of positive numbers δ and η such that $\delta \leq \eta < \epsilon$ there corresponds a sequence $\{\alpha_i\}$ of arcs of R with the following properties:* 1) *for each i, α_i is an arc of $\bar{S}(P, \delta)$ from a point of $F(P, \delta)$ to a point P_i;* 2) *$\{P_i\}$ converges to P; and* 3) *no two different elements of $\{\alpha_i\}$ can be joined by an arc of $R \cdot S(P, \eta)$.*

By a theorem of G. T. Whyburn,[5] we can find a positive number ϵ and a sequence $\{P_i\}$ of points of R having P as a sequential limit point and such that no two of these points can be joined by an arc of R which is of diameter $< 2\epsilon$. Let δ and η be any two positive numbers such that $\delta \leq \eta < \epsilon$. Without loss of generality we may suppose that all of the points of $\{P_i\}$ are contained in $S(P, \delta)$. Since R is arcwise connected, we can find an arc of R from P_1 to P_2, which must be of diameter $> 2\epsilon$, and hence cannot be contained in $S(P, \epsilon)$. It follows that R has a point Q not contained in $S(P, \epsilon)$. For each i, construct an arc P_iQ of R from P_i to Q, and on that arc let Q_i be the first point from P_i belonging to $F(P, \delta)$. Let α_i denote the subarc of P_iQ from P_i to Q_i. We show that the sequence $\{\alpha_i\}$ fulfills the requirements of the lemma. That conditions 1) and 2) are fulfilled is evident. If 3) were not fulfilled, there would exist an arc β of $R \cdot S(P, \eta)$ joining two different arcs of $\{\alpha_i\}$, say α_m and α_n. Then in $\beta + \alpha_m + \alpha_n$ we could construct an arc of R of diameter $< 2\epsilon$ joining two distinct points P_m and P_n of $\{P_i\}$, contrary to the selection of $\{P_i\}$.

LEMMA 2. *If P is an endpoint of an arc t of a simple closed curve J of a connected space M with the property that the complement of every simple closed curve of M consists of not more than a finite number of components, then there exists a positive number δ such that, if $\langle a \rangle$ is an arc of $M \cdot S(P, \delta) - J$ both of whose endpoints are in $\langle t \rangle$, then no component of $M - (J + a)$ has its boundary contained entirely in a.*

Supposing the contrary, we let δ_1 be any positive number and let $\langle a_1 \rangle$ be an arc of $M \cdot S(P, \delta_1) - J$ whose endpoints are contained in $\langle t \rangle$ such that there is a component R_1 of $M - (J + a_1)$ whose boundary is contained entirely in a_1. We let δ_2 be a positive number such that $\delta_2 < \frac{1}{2}\delta_1$, $\delta_2 < \rho(P, a_1)$,

[5] "Concerning Menger regular curves," *Fundamenta Mathematicae*, vol. 12 (1928), pp. 264-294; Theorem 3, p. 272.

and $t \cdot S(P, \delta_2)$ is contained in a connected subset of $t - a_1$. Select an arc $\langle a_2 \rangle$ of $M \cdot S(P, \delta_2) - J$ whose endpoints are in $\langle t \rangle$ such that there is a component R_2 of $M - (J + a_2)$ whose boundary is contained entirely in a_2. Continuing inductively, we define a sequence $\{\delta_i\}$ of positive numbers, a sequence $\{a_i\}$ of arcs whose endpoints are in $\langle t \rangle$, and a sequence $\{R_i\}$ of sets such that, for each i, 1) $\delta_i < \frac{1}{2}\delta_{i-1}$; 2) $\delta_i < \rho(P, a_{i-1})$; 3) $t \cdot S(P, \delta_i)$ is contained in a connected subset of $t - a_{i-1}$; 4) $\langle a_i \rangle \subset M \cdot S(P, \delta_i) - J$; and 5) R_i is a component of $M - (J + a_i)$ such that $F(R_i) \subset a_i$. From 2) and 4) it follows that 6) $a_i \cdot a_j = 0$ for $i \neq j$. From 1) and 4) it follows that 7) $\{a_i\}$ converges to P. If for each i we denote by b_i the set of points of t between the endpoints of a_i, it follows from 2), 3), and 4) that 8) $b_i \cdot b_j = 0$ for $i \neq j$. From 6), 7), and 8) it follows that $t + \Sigma a_i - \Sigma b_i$ is an arc t'. From 5) and 6) it follows that the sets R_i are all distinct. They are all components of the complement of the simple closed curve $t' + (J - t)$. This contradiction of our hypothesis that every simple closed curve of M has at most a finite number of complementary domains completes the proof of the lemma.

LEMMA 3. *Let M be a Peano space with the property that the complement on M of every simple closed curve of M consists of at least two and at most a finite number of components. For no boundary point P of a complementary domain D of a simple closed curve J of M does there exist a positive number ϵ such that to every pair of positive numbers δ and η such that $\delta \leq \eta < \epsilon$ there corresponds a sequence $\{\alpha_i\}$ of arcs of D with the following properties: 1) for each i, α_i is an arc of $\bar{S}(P, \delta)$ from a point of $F(P, \delta)$ to a point P_i; 2) $\{P_i\}$ converges to P; and 3) no two different elements of $\{\alpha_i\}$ can be joined by an arc of $D \cdot S(P, \eta)$.*

Suppose the contrary, that there is a boundary point P of a complementary domain D of a simple closed curve J of M and a positive number ϵ such that to every pair of positive numbers δ and η such that $\delta \leq \eta < \epsilon$ there corresponds a sequence $\{\alpha_i\}$ of arcs of D with the properties listed above.

Let P' be a particular point of J distinct from P. Let η be a positive number such that $\eta < \epsilon$ and $\eta < \rho(P, P')$.

Let $D_1, D_2, \cdots, D_k$ denote the components of $M - J$ distinct from D, where D_i has a boundary point distinct from P for $i = 1, 2, \cdots, k'$, but not for $i = k' + 1, \cdots, k$. For $i = 1, 2, \cdots, k'$, let Q_i be a particular boundary point of D_i distinct from P.

If P is not the only boundary point of D (we shall prove that it is not later), then, since P is a non-cut point of J and D is connected, it follows that P is a non-cut point of $J + D$. Furthermore, $J + D$ is a Peano space. Hence

we can find a positive number δ' such that $J + D - S(P, \eta)$ is contained in a connected subset of $J + D - S(P, \delta')$.[6]

By means of lemma 2 we determine a positive number δ'' such that, for t either arc of J from P to P', if $\langle a \rangle$ is an arc of $M \cdot S(P, \delta'') - J$ whose endpoints are in $\langle t \rangle$, then no component of $M - (J + a)$ has its boundary contained entirely in a.

Let δ''' be a positive number such that $M \cdot \bar{S}(P, \delta''')$ is compact.

Let δ be any positive number less than each of the numbers η, δ', δ'', δ''', and $\rho(P, \sum_{i=1}^{k'} Q_i)$. Should the number δ' as defined above fail to exist, we have here simply to omit the symbol δ'.

With this selection of η and δ, we determine a sequence $\{\alpha_i\}$ of arcs of D such that 1) for each i, α_i is an arc of $\bar{S}(P, \delta)$ from a point of $F(P, \delta)$ to a point P_i; 2) $\{P_i\}$ converges to P; and 3) no two different elements of $\{\alpha_i\}$ can be joined by an arc of $D \cdot S(P, \eta)$. The sequence $\{\alpha_i\}$ will contain a convergent subsequence, and this convergent subsequence has all of the properties we have listed for $\{\alpha_i\}$; hence without loss of generality we may suppose $\{\alpha_i\}$ to converge. Denote by H the sequential limit of $\{\alpha_i\}$. Since $\delta < \delta'''$ and, for each i, $P_i \subset \alpha_i \subset \bar{S}(P, \delta)$, $\alpha_i \cdot F(P, \delta) \neq 0$, and α_i is connected, it follows that $P \subset H \subset \bar{S}(P, \delta)$, $H \cdot F(P, \delta) \neq 0$, and H is connected.

We show that $H \subset J$. If not, then there is a point x of H in $D \cdot \bar{S}(P, \delta)$. Since M is locally connected, we can find a number $\xi > 0$ such that every point of $M \cdot S(x, \xi)$ can be joined to x by an arc of $M \cdot [x, \rho(x, J + F(P, \eta))]$. Since x is in H, the limit set of $\{\alpha_i\}$, we can find two distinct integers m and n such that α_m and α_n meet $S(x, \xi)$; let y and z be points of $\alpha_m \cdot S(x, \xi)$ and $\alpha_n \cdot S(x, \xi)$ respectively. Let yx and zx be arcs of $M \cdot S[x, \rho(x, J + F(P, \eta))]$ joining y and z respectively to x. In $yx + zx$ we can find an arc f joining α_m and α_n. Since $f \subset M \cdot S[x, \rho(x, J)]$, we have $f \subset D$. Since $x \subset S(P, \eta)$ and $yx + zx \subset S[x, \rho(x, F(P, \eta))]$, we have $f \subset yx + zx \subset S(P, \eta)$. Thus f is an arc of $D \cdot S(P, \eta)$ joining the two different elements α_m and α_n of $\{\alpha_i\}$. This contradicts our choice of $\{\alpha_i\}$. We conclude that $H \subset J$.

Since every point of H is a limit point of $\sum_{i=1}^{\infty} \alpha_i \subset D$ and $H \subset J$, it follows that every point of H is a boundary point of D. We have shown that $H \cdot F(P, \delta) \neq 0$, and thus P is not the only boundary point of D. This establishes the existence of the number δ' as defined above.

---

[6] R. L. Wilder, "The sphere in topology," *American Mathematical Society Semicentennial Publications*, vol. II: Semicentennial Addresses, pp. 136-184; Corollary to Theorem 27, p. 148.

The set H, being a connected subset of J and containing P and a point of $F(P,\delta)$, contains an arc h from P to a point of $F(P,\delta)$ such that $\langle h\rangle \subset S(P,\delta)$.

If e is any open subarc of h, there exists an integer n such that, for $i \geqq n$, α_i can be joined to e by an arc of $S(P,\delta)$ lying in D except for its endpoint in e. To obtain such a number n we proceed in the following way. Let x be any point of e. Since M is locally connected, we can find a number $\xi > 0$ such that every point of $M \cdot S(x,\xi)$ can be joined to x by an arc of $M \cdot S[x, \rho(x, M \cdot F(P,\delta) + J - e)]$. Since x is a point of the limit set of $\{\alpha_i\}$, we can find an integer n such that $\rho(x,\alpha_i) < \xi$ for $i \geq n$. We show that this n suffices. For $i \geq n$ we can find an arc $y_i x$ of $M \cdot S[x, \rho(x, M \cdot F(P,\delta) + J - e)]$ from a point y_i of $\alpha_i \cdot S(x,\xi)$ to x. The subarc $u_i v_i$ of this arc $y_i x$ from the last point u_i which belongs to α_i to the first point v_i after u_i which belongs to e satisfies the requirements; for $u_i v_i \cdot J = v_i$, since $u_i v_i \subset S[x, \rho(x, J - e)]$, and hence $u_i v_i\rangle \subset D$ since $u_i \subset \alpha_i \subset D$; and also $u_i v_i \subset S(P,\delta)$ since $x \subset \langle h\rangle \subset S(P,\delta)$ and $u_i v_i \subset S[x, \rho(x, M \cdot F(P,\delta))]$.

Let e_1, e_2, e_3, and e_4 be mutually exclusive open subarcs of h, numbered in order from P. By the previous paragraph, for $j = 1, 2, 3, 4$, we can find an integer n_j such that, for $i \geq n_j$, there exists an arc $u_{ij} v_{ij}$ joining α_i to e_j and lying in $D \cdot S(P,\delta)$ except for v_{ij}. Let i_0 and i_1 be two different integers such that $i_0 \geq n_1 + n_3$ and $i_1 \geq n_2 + n_4$. In $\alpha_{i_1} + u_{i_12} v_{i_12} + u_{i_14} v_{i_14}$ we construct an arc θ spanning h. The endpoints of θ are v_{i_12} and v_{i_14}; these we rename w and z. We have $w \subset e_2$ and $z \subset e_4$.

For any two points a and b on h the symbol ab will denote the arc of h from a to b.

We observe that h is a subarc of one of the arcs of J from P to P' Hence, since $\langle\theta\rangle \subset D \cdot S(P,\delta)$ while $\delta < \delta''$, it follows that no component of $D - \theta$ has its boundary contained entirely in θ. Thus every component of $D - \theta$ has a boundary point in $J - (w + z)$. Let $G_1, G_2, \cdots, G_r$ be the components of $D - \theta$ having their boundaries contained in $\theta + wz$. That there can be only a finite number of these follows from the fact that each is a component of the complement of the simple closed curve $\theta + wz$. For $i = 1, 2, \cdots, r$, let g_i be a particular boundary point of G_i in $\langle wz\rangle$.

Following closely the process used in obtaining θ, we determine an integer i_2 different from i_0 and i_1 such that α_{i_2} can be joined to each of the open arcs $e_2 \cdot \prod_{i=1}^{r} \langle wg_i\rangle$ and $e_4 \cdot \prod_{i=1}^{r} \langle g_i z\rangle$ by an arc lying in $D \cdot S(P,\delta)$ except for one endpoint. In case $r = 0$, we have simply to omit the smybols $\prod_{i=1}^{r} \langle wg_i\rangle$ and $\prod_{i=1}^{r} \langle g_i z\rangle$. From the sum of α_{i_2} and two such joining arcs we construct an

arc λ spanning h and having its endpoints x and y in $e_2 \cdot \prod_{i=1}^{r} \langle wg_i \rangle$ and $e_4 \cdot \prod_{i=1}^{r} \langle g_i z \rangle$ respectively. Since each component of $D - \lambda$ whose boundary is contained in $\lambda + (J - xy)$ is a component of the complement of this simple closed curve, it follows that the number of these is finite: let them be $E_1, E_2, \cdots, E_s$. Since $\langle \lambda \rangle \subset D \cdot S(P, \delta)$ and $\delta < \delta''$, it follows that every other component of $D - \lambda$ must have a boundary point in $\langle xy \rangle$.

It follows immediately from the construction process that θ and λ can be joined to α_{i_1} and α_{i_2} respectively by arcs of $D \cdot S(P, \delta)$. Also, $\langle \theta \rangle + \langle \lambda \rangle \subset D \cdot S(P, \delta)$. The endpoints of θ and λ are all distinct. Hence if θ and λ were to intersect, they would intersect in an interior point of both, and in their sum together with arcs of $D \cdot S(P, \delta)$ joining them to α_{i_1} and α_{i_2} we could construct an arc of $D \cdot S(P, \delta) \subset D \cdot S(P, \eta)$ joining α_{i_1} and α_{i_2}, contrary to the construction of $\{\alpha_i\}$. We conclude that $\theta \cdot \lambda = 0$.

Construct the simple closed curve $J' = \theta + \lambda + wx + yz$. Then $J \cdot J' = wx + yz \subset e_2 + e_4$. Let K denote the component of $M - J'$ which contains α_{i_0}. Then K must contain $u_{i_0 j} v_{i_0 j}$ for $j = 1$ and 3; for if any of these arcs met J', it would have to meet $\langle \theta \rangle + \langle \lambda \rangle$, and hence we could construct an arc of $D \cdot S(P, \delta)$ joining α_{i_0} to either α_{i_1} or α_{i_2}. Hence K must contain $e_1 + e_3$, and hence, in turn, $J - J'$. In particular, $K \supset P + P' + \sum_{i=1}^{k'} Q_i$. It follows that $K \supset \sum_{i=1}^{k} D_i$; for $i = 1, 2, \cdots, k'$, Q_i is a limit point of D_i contained in K, and, for $i = k' + 1, \cdots, , k$, P is a limit point of D_i contained in K.

No component of $M - J'$ different from K can have boundary points in both $\langle \theta \rangle$ and $\langle \lambda \rangle$. To show this, we suppose L to be a component of $M - J'$ different from K having boundary points in both $\langle \theta \rangle$ and $\langle \lambda \rangle$. Then we can construct an arc μ from a point of $\langle \theta \rangle$ to a point of $\langle \lambda \rangle$ lying in L except for its endpoints. Since $K \supset \sum_{i=1}^{k} D_i + (J - J')$, we have $L \subset D$, and thus $\mu \subset D$. Then in $\mu + \alpha_{i_1} + \alpha_{i_2}$ and the joining arcs used in constructing θ and λ we can construct an arc of D joining α_{i_1} and α_{i_2}. Hence, by the construction of $\{\alpha_i\}$, this arc cannot be contained in $S(P, \eta)$. Let Q be a point of $\mu \cdot [D - S(P, \eta)]$. By definition of δ', the two points P' and Q, belonging to $J + D - S(P, \eta)$, are contained in a connected subset of $J + D - S(P, \delta')$ $\subset J + D - S(P, \delta) \subset J + D - J' \subset M - J'$. Since $P' \subset K$ and $Q \subset \mu \subset L$, this contradicts our supposition that K and L were different components of $M - J'$.

We now consider the set $D - J'$. Certain components of this set may

have boundary points in both $\langle\theta\rangle$ and $\langle\lambda\rangle$, and hence must belong to K, by the preceding paragraph; every other component of $D - J'$ has boundary points in $\theta + J$ only or in $\lambda + J$ only, and hence is a component of $D - \theta$ not meeting λ or a component of $D - \lambda$ not meeting θ. We have $\sum_{i=1}^{r} g_i \subset \langle xy\rangle \subset K$, and thus $\sum_{i=1}^{r} G_i \subset K$. Every other component of $D - \theta$, as we have already seen, has a boundary point in $J - wz \subset K$, and hence those not meeting λ are contained in K. All components of $D - \lambda$ other than $E_1, E_2, \cdots, E_s$ have boundary points in $\langle xy\rangle \subset K$, and hence those which do not meet θ are contained in K. We may suppose the sets $E_1, E_2, \cdots, E_s$ to be numbered so that, for s' properly chosen, $F(E_i) \cdot (J - wz) = 0$ for $i = 1, 2, \cdots, s'$ and $F(E_i) \cdot (J - wz) \neq 0$ for $i = s' + 1, \cdots, s$. Then $\sum_{i=s'+1}^{s} E_i \subset K$.

We have shown that all of $M - J'$ is contained in K except $\sum_{i=1}^{s'} E_i$. Since, by hypothesis, J' must separate M, it follows that $\sum_{i=1}^{s'} E_i \neq 0$, and thus $s' \geq 1$. Let $E = E_1$. Since $F(E) \subset \lambda + (J - xy)$ while $F(E) \cdot (J - wz) = 0$, we have $F(E) \subset \lambda + wx + yz$.

Thus for δ any positive number sufficiently small, we have shown how to determine points w, x, y, and z distinct from each other and from P on J in the order $Pwxyz$ and such that the arc $Pwxyz$ of J is contained in $S(P, \delta)$; an arc λ from x to y such that $\langle\lambda\rangle \subset D \cdot S(P, \delta)$; and a component E of $D - \lambda$ such that $F(E) \subset \lambda + wx + yz$.

Let δ_1 be a particular positive number satisfying the conditions on δ. Determine w_1, x_1, y_1, z_1, λ_1, and E_1 such that w_1, x_1, y_1, and z_1 are points on J distinct from each other and from P in the order $Pw_1x_1y_1z_1$ and such that the arc $Pw_1x_1y_1z_1$ of J is contained in $S(P, \delta_1)$; λ_1 is an arc from x_1 to y_1 such that $\langle\lambda_1\rangle \subset D \cdot S(P, \delta_1)$; and E_1 is a component of $D - \lambda_1$ such that $F(E_1) \subset \lambda_1 + w_1x_1 + y_1z_1$. Since $\rho(P, \lambda_1 + w_1x_1y_1z_1) > 0$, we can select a positive number δ_2 such that $\delta_2 < \rho(P, \lambda_1 + w_1x_1y_1z_1)$ and $\delta_2 < \frac{1}{2}\delta_1$. Corresponding to δ_2, we obtain w_2, x_2, y_2, z_2, λ_2, and E_2. Then we select a positive number δ_3 such that $\delta_3 < \rho(P, \lambda_2 + w_2x_2y_2z_2)$ and $\delta_3 < \frac{1}{2}\delta_2$. Continuing inductively, we determine sequences $\{w_i\}$, $\{x_i\}$, $\{y_i\}$, $\{z_i\}$, $\{\lambda_i\}$, $\{E_i\}$, and $\{\delta_i\}$ such that, for each i, 1) w_i, x_i, y_i, and z_i are points on J distinct from each other and from P in the order $Pw_ix_iy_iz_i$; 2) the arc $Pw_ix_iy_iz_i$ of J is contained in $S(P, \delta_i)$; 3) λ_i is an arc from x_i to y_i such that $\langle\lambda_i\rangle \subset D \cdot S(P, \delta_i)$; 4) E_i is a component of $D - \lambda_i$ such that $F(E_i) \subset \lambda_i + w_ix_i + y_iz_i$; 5) $\delta_i < \frac{1}{2}\delta_{i-1}$; and 6) $\delta_i < \rho(P, \delta_{i-1}$

$+ w_{i-1}x_{i-1}y_{i-1}z_{i-1})$. From 3) and 6) it follows that 7) $\lambda_i \cdot \lambda_j = 0$ if $i \neq j$. From 3) and 5) it follows that 8) $\{\lambda_i\}$ converges to P. From 2) and 6) it follows that 9) $w_i x_i y_i z_i \cdot w_j x_j y_j z_j = 0$ for $i \neq j$. It readily follows that $(J - \sum_{i=1}^{\infty} x_i y_i) + \sum_{i=1}^{\infty} \lambda_i$ is a simple closed curve J_0. We have $J_0 \supset \sum_{i=1}^{\infty} (\lambda_i + w_i x_i + y_i z_i)$. Hence, from 4), E_i is a complementary domain of J_0 for every i. It follows from 7) and 9) that E_i and E_j are distinct for $i \neq j$. Thus J_0 has infinitely many complementary domains, contrary to our hypothesis. This completes the proof of Lemma 3.

THEOREM 1. ACCESSIBILITY THEOREM. *Let M be a Peano space with the property that the complement on M of every simple closed curve of M consists of at least two and at most a finite number of components, and let D be a complementary domain of a simple closed curve J of M. Then every point of the boundary of D is regularly accessible from D.*

This theorem is an immediate consequence of Lemma 1 and Lemma 3. Suppose D has a boundary point P which is not regularly accessible from D. Then, by Lemma 1 there exists a positive number ϵ such that to every pair of positive numbers δ and η such that $\delta \leq \eta < \epsilon$ there corresponds a sequence $\{\alpha_i\}$ of arcs of D with the following properties: 1) for each i, α_i is an arc of $\bar{S}(P, \delta)$ from a point of $F(P, \delta)$ to a point P_i; 2) $\{P_i\}$ converges to P; and 3) no two different elements of $\{\alpha_i\}$ can be joined by an arc of $D \cdot S(P, \eta)$. By Lemma 3, however, this is impossible.

This accessibility theorem is used in proving the following theorem, which, as has already been pointed out, is a result obtained previously by Zippin by a different method.

THEOREM 2. *Let M be a compact Peano space such that* 1) *M is not separated by any pair of its points and* 2) *the complement on M of every simple closed curve of M consists of exactly two components. Then M is a simple closed surface.*

We suppose that there exists a compact Peano space M which satisfies the hypothesis of the theorem and yet fails to be a simple closed surface. Since M has no cut point, it must contain a simple closed curve. From a theorem of Zippin [7] it follows that M must fail to satisfy the Jordan Curve Theorem; that is, there must exist a simple closed curve J in M such that one component D of $M - J$ has as its boundary a proper subset of J. Let t be a minimal arc of J containing the boundary of D, and let P and Q denote the

---

[7] *Loc. cit.*, Theorem 3′, p. 340.

endpoints of t. We denote by D' the other component of $M - J$ and by t' the arc $J - \langle t \rangle$.

From the accessibility theorem it follows that both P and Q are accessible from D. We construct an arc f from P to Q such that $\langle f \rangle \subset D$.

Suppose that f separates D: $D - f = D_1 + D_2$, where $D_1 \neq 0 \neq D_2$ and neither D_1 nor D_2 contains a limit point of the other. Consider the simple closed curve $J_1 = t + f$. We have $F(D_1) \subset J_1$, and thus $D_1 + D_2$ consists of at least two different domains of $M - J_1$. The component of $M - J_1$ containing D' thus makes at least a third component of $M - J_1$. This contradicts the hypothesis that every simple closed curve of M has exactly two complementary domains. Hence $D - f$ must be connected.

Since by hypothesis the two points P and Q cannot separate D from $\langle t \rangle + D' + \langle t' \rangle$, it follows that $\langle t \rangle$ must contain a boundary point of D. Likewise, since P and Q cannot separate $D + \langle t \rangle$ from $D' + \langle t' \rangle$, we have that $\langle t \rangle$ contains a boundary point of D'. Now consider the simple closed curve $J_2 = f + t'$. The component of $M - J_2$ which contains $\langle t \rangle$ must contain D' and also the connected set $D - f$. This accounts for all of M, and thus the simple closed curve J_2 fails to separate M. This contradiction of our hypothesis completes the proof of the theorem.

HAVERFORD COLLEGE.

A DISCRETE GROUP ARISING IN THE STUDY OF DIFFERENTIAL OPERATORS.* [1]

By Earl D. Rainville.

1. Introduction. The determination of the behavior of a certain class of linear differential equations under the application of the Laplace integral transformation is considerably facilitated by the introduction of another operator [2] σ and the study of the behavior of linear differential operators when subjected to σ.

We redefine σ for convenience of reference. Let $D \equiv d/dx$ be the usual symbol for differentiation with respect to x and let $D^0 \equiv 1$, the identity operator. We shall be concerned with only those linear differential operators which are polynomials in D and x. We define σ as a linear operator such that

$$\sigma x^k D^n = (-1)^k D^k x^n, \tag{1}$$

where k and n are non-negative integers.[3] By the linearity of σ we mean that

$$\sigma[\sum_s b_s x^{k_s} D^{n_s}] = \sum_s b_s \sigma(x^{k_s} D^{n_s}), \tag{2}$$

where k_s, n_s, are non-negative integers and the b_s are any complex constants.

We next define [4] another operator α by

$$\alpha x^k D^n = (-1)^n D^n x^k, \tag{3}$$

where k and n are non-negative integers, and the requirement that α be linear in the sense of (2) above. When α is applied to a linear differential operator

* Received June 13, 1940.

[1] Presented to the Society under a different title, April 15, 1939. The author wishes to express his indebtedness to G. Y. Rainich and R. M. Thrall for suggestions regarding the presentation of this material.

[2] The operator σ has interesting properties some of them known to be of importance in applied mathematics. For the behavior of differential operators under σ see E. D. Rainville, "Linear differential invariance under an operator related to the Laplace transformation, *American Journal of Mathematics*, vol. 62 (1940), pp. 391-405. For the precise relation of σ to the Laplace Transformation see Enzo Levi, "Proprietà caratteristische della trasformazione di Laplace," *Rendiconti Accademia Lincei* (6), vol. 24, (1936), pp. 422-426.

[3] By $D^k x^n$ we mean that differential operator which acting upon an indeterminate function F yields the k-th derivative of the product $x^n F$.

[4] For an elementary discussion of α see E. D. Rainville, "Adjoints of linear differential operators," *American Mathematical Monthly*, vol. 46 (1939), pp. 623-627.

of the type under consideration, it produces the classical adjoint of that operator.

Finally, in analogy to the above definitions, we introduce two other linear operators ψ and μ such that

$$\psi x^k D^n = D^k x^n, \tag{4}$$

$$\mu x^k D^n = D^n x^k, \tag{5}$$

where k and n are non-negative integers.

2. Results. Each of the operators σ, α, ψ, μ, may be expressed as a product of the other three. Exhibiting the inverse of each, we show that these operators generate a non-Abelian group, which may be taken to be $G\{\sigma, \alpha, \psi\}$. The elements of G may be represented uniquely in a simple form (Theorem 4). We determine five sole irredundant defining relations for G.

3. Preliminary Theorems. Let $E = \sigma^0 = \alpha^0 = \psi^0 = \mu^0$ be the identity operator. It is convenient to use the known results $\sigma^4 = E$, $\alpha^2 = E$, and

$$\sigma^2 x^k D^n = (-1)^{k+n} x^k D^n. \tag{6}$$

We see at once that the inverses of σ and α exist.

There exists no positive integral power of ψ equal to E and the same remark holds for μ. To see this consider the differential operator $B = xD$. Now $\psi B = Dx = xD + 1 = B + 1$. Hence $\psi^s B = B + s$ for $s \geq 0$. We can go further than this. Suppose we define the sum of $g_1 \epsilon G$ and $g_2 \epsilon G$ as that element of G which operating on any linear differential operator A yields the differential operator $(g_1 A + g_2 A)$, then we may speak of the ring generated by σ, α, and ψ. In the linear set over this ring with coefficients in the complex domain there is no identically vanishing polynomial in ψ alone except that with all coefficients zero; that is, in this extended sense there is no counterpart of the relations $\sigma^4 - E = 0$ and $\alpha^2 - E = 0$.

We are able to construct the inverses of ψ and μ with the aid of σ and α. For the sake of symmetry we include results unnecessary for our immediate purpose.[5]

THEOREM 1. $(\alpha\sigma)^2 = (\sigma\alpha)^2 = (\alpha\mu)^2 = (\mu\alpha)^2 = (\sigma\psi)^2 = (\psi\sigma)^2 = E$.

Because of the linearity of the operators involved, we need verify the results only when the operators act upon $x^k D^n$. Directly from the definitions we see that

[5] What is essentially part of Theorem 1 appears in L. Schlesinger, *Handbuch der Linearen Differentialgleichungen*, Leipzig, vol. 1 (1895), p. 426.

(7) $$\alpha\sigma x^k D^n = \alpha(-1)^k D^k x^n = \alpha^2 x^n D^k = x^n D^k,$$

and similarly $\sigma\alpha x^k D^n = (-1)^{k+n} x^n D^k$, $\alpha\mu x^k D^n = (-1)^n x^k D^n$, $\sigma\psi = \alpha\mu$. Theorem 1 follows at once. We have incidentally shown that $\sigma\psi\sigma$ and $\alpha\mu\alpha$ are respectively the inverses of ψ and μ.

THEOREM 2. $\sigma = \mu\alpha\psi$, $\alpha = \mu\sigma\psi$, $\psi = \mu\alpha\sigma$, $\mu = \psi\alpha\sigma$.

The proof of Theorem 2 may be made similarly to that of Theorem 1. As an example, using (7) we find that

$$\mu\alpha\sigma x^k D^n = \mu x^n D^k = D^k x^n = \psi x^k D^n.$$

Finally we have in the same manner

THEOREM 3. $(\alpha\psi)^2 = (\psi\alpha)^2 = (\sigma\mu)^2 = (\mu\sigma)^2 = \sigma^2$.

Part of Theorem 3 will be used as a defining relation for G.

4. The Group $G\{\sigma, \alpha, \psi\}$. Consider first the group T_8 generated by σ and α. This is simply isomorphic with the octic group [6] with defining relations $\sigma^4 = \alpha^2 = (\alpha\sigma)^2 = E$. The elements of T_8 may be taken to be E, σ, σ^2, σ^3, α, $\alpha\sigma$, $\alpha\sigma^2$, $\alpha\sigma^3$.

From Theorem 1 we have $\sigma\alpha\sigma = \alpha$. Hence $\sigma^2\alpha = \sigma^3\alpha\sigma = \sigma^4\alpha\sigma^2 = \alpha\sigma^2$. Since σ^2 is also commutative with σ, we have proved that, if $t \in T_8$, then $\sigma^2 t = t\sigma^2$.

We already know from Theorem 3 that $\psi\alpha\psi\alpha = \sigma^2 = \alpha\psi\alpha\psi$. Then from the identity $\psi\alpha\psi\alpha\psi = \psi\alpha\psi\alpha\psi$ we may conclude that $\sigma^2\psi = \psi\sigma^2$. Next by combining various results given alone, we have

$$\alpha\sigma\psi = \alpha\sigma\psi\sigma\sigma^2\sigma = \alpha\sigma\psi\sigma\psi\alpha\psi\alpha\sigma = \alpha E\alpha\psi\alpha\sigma = \psi\alpha\sigma.$$

Thus we have proved that ψ is commutative with σ^2 and with $\alpha\sigma$.

Now let G be the group generated by σ, α, and ψ. We shall prove

THEOREM 4. *If $g \in G\{\sigma, \alpha, \psi\}$, then g may be represented in just one of the two forms $\psi^s t$ or $\sigma\psi^{s+1} t$, where $s \geqq 0$ and $t \in T_8$, and this representation is unique.*

The group T_8 contains a subgroup H generated by σ^2 and $\alpha\sigma$. T_8 may be decomposed into $H + \sigma H$. Each element of H is commutative with ψ. Next. by Theorem 1, $\psi\sigma\psi = \sigma^3$, so that if σ does occur between two powers of ψ. one or both exponents of ψ may be brought to zero. Hence, if $g \in G$ and $g \setminus T$,

---

[6] See Carmichael, *Groups of Finite Order*, 1937, p. 176.

then g may be put in the form $t_1\psi^s t_2$, where $t_1, t_2 \epsilon T$ and $s > 0$. Again, t_1 is either commutative with ψ or is the product of σ and an element commutative with ψ. We have shown the existence of the representation of Theorem 4.

Let us turn to the question of uniqueness. Let s_1 be the smallest positive integer for which there exists an equality of the type $\delta_1\psi^{s_1}t_1 = \delta_2\psi^{s_2}t_2$ in which $\delta_1, \delta_2 = E$ or σ and $t_1, t_2 \epsilon T_8$ and in which not both of the equations $\delta_1 = \delta_2$ and $t_1 = t_2$ hold. Then

$$\psi^{s_1} = \delta_1^{-1}\delta_2\psi^{s_2}t_2t_1^{-1} = \delta_3\psi^{s_2}t_3, \tag{8}$$

where $\delta_3 = E$ or σ and $t_3 \epsilon T_8$ and not both of $\delta_3 = E$ and $t_3 = E$ hold. We now apply the inverse of ψ and find

$$\psi^{s_1-1} = \sigma\psi\sigma\delta_3\psi^{s_2}t_3. \tag{9}$$

If $\delta_3 = E$, then

$$\psi^{s_1-1} = \psi^{s_2-1}t_3 \tag{10}$$

where $t_3 \neq E$. If $\delta_3 = \sigma$,

$$\psi^{s_1-1} = \sigma\psi^{s_2+1}t_3, \tag{11}$$

so that in any case ψ^{s_1-1} is of the form $\delta_4\psi^{s_3}t_4$ with not both $\delta_4 = E$ and $t_4 = E$. Since s_1 was to be a minimum, this is a contradiction for $s_1 > 1$. If $s_1 = 1$. then from (10) or (11) we get an equation of the type $\psi^s = t_5$ where $t_5 \epsilon T_8$ and not both $s = 0$ and $t_5 = E$ hold. Then $t_5^4 = E$ but $\psi^{4s} \neq E$, hence the representation of Theorem 4 is unique.

5. The Defining Relations. Consider now an abstract group with three generators $\beta_1, \beta_2, \beta_3$, and the sole defining relations

$$\beta_1^4 = \beta_2^2 = (\beta_1\beta_2)^2 = (\beta_1\beta_3)^2 = E, \ (\beta_2\beta_3)^2 = \beta_1^2. \tag{12}$$

We see that, if we put $\beta_1 = \sigma$, $\beta_2 = \alpha$, $\beta_3 = \psi$, the relations of (12) are satisfied. The proof of Theorem 4 was based upon only the corresponding relations

$$\sigma^4 = \alpha^2 = (\sigma\alpha)^2 = (\sigma\psi)^2 = E, \qquad (\alpha\psi)^2 = \sigma^2, \tag{13}$$

and the fact that $\psi^s \neq E$ for $s > 0$. Any identity involving the elements of G must then be a result of (13). We have shown that G is simply isomorphic with the abstract group defined above.

In order to assure ourselves that the defining relations (12) are irredundant, we make certain other selections for $\beta_1, \beta_2, \beta_3$. It is easily verified that if we choose $\beta_1 = \sigma, \alpha\sigma\psi, \alpha\sigma, \sigma, \sigma$; $\beta_2 = \psi, \alpha, \alpha, \alpha\sigma, \alpha\sigma$; $\beta_3 = \alpha, \sigma\psi, E, \sigma^3\psi, \sigma^2\psi$, respectively, then in each of the five cases one, and only one, of the relations in (12) breaks down. These choices show that the relations (12) are irredundant.

The operators σ^2, $\alpha\sigma$, and ψ generate an infinite Abelian subgroup of G. We call this subgroup N and note that its representative element may be written in the form $\psi^m h$ where $h \in H$, the group of order 4 generated by σ^2 and $\alpha\sigma$, and where m is an integer, positive, negative, or zero. We shall see that G may be decomposed into $N + \sigma N$. Indeed, from Theorem 1 we have $\psi\sigma = \sigma^3\psi^{-1}$ which, in view of the commutivity of ψ and σ^2 becomes $\psi\sigma = \sigma\psi^{-1}\sigma^2$. Hence $\psi^2\sigma = \psi\sigma\psi^{-1}\sigma^2 = \sigma\psi^{-1}\sigma^2\psi^{-1}\sigma^2 = \sigma\psi^{-2}\sigma^4$. Repeated applications of this process lead to

$$\psi^s\sigma = \sigma\psi^{-s}\sigma^{2s}, \tag{14}$$

where s is a positive integer. Theorem 4 shows that any element of G may be written in one of the four forms $\psi^s h$, $\psi^s\sigma h$, $\sigma\psi^s h$, $\sigma\psi^s\sigma h$, with $s \geq 0$, $h \in H$. Using (14) we see that $\psi^s\sigma h = \sigma\psi^{-s}\sigma^{2s}h$ and $\sigma\psi^s\sigma h = \sigma^2\psi^{-s}\sigma^{2s}h = \psi^{-s}\sigma^{2s+2}h$. Hence $\psi^s h$ and $\sigma\psi^s\sigma h$ are elements of N and $\psi^s\sigma h$ and $\sigma\psi^s h$ are products of σ with elements of N.

UNIVERSITY OF MICHIGAN.

THE DIRICHLET PROBLEM FOR A HYPERBOLIC EQUATION.*

By Fritz John.

This paper deals with the Dirichlet problem for the hyperbolic differential equation

$$u_{xy} = 0, \tag{1}$$

i. e. with the problem of determining a solution u of (1) from given values on a closed curve C.[1] Simple examples show that the Dirichlet problem for a hyperbolic equation has a completely different character from that of the corresponding problem for an elliptic equation such as the potential equation

$$u_{xx} + u_{yy} = 0. \tag{2}$$

In the case of a hyperbolic equation the Dirichlet problem certainly is not a "natural"[2] problem of mathematical physics, as its solution may neither exist, nor be uniquely determined, nor depend continuously on the data.

However it is possible to obtain fairly general positive results in this connection. The general solution of (1) is well known to be of the form $u = f(x) + g(y)$; more exactly, a solution of (1) is of that form with univalued functions f and g in every region which is convex in the x- and y-direction, i. e. the boundary C of which is a Jordan curve, intersected in at most two points by every parallel to the x- or y-axis. We restrict ourselves to regions of this kind.

We call the points of C, in which there is a line of support parallel to the x- or y-axis, the "vertices" of C.[3] It is evident, that C has either two, three, or four vertices, the last alternative representing the general case. The behavior of the Dirichlet problem for C depends largely on that of certain transformations of C into itself, which suggest themselves immediately. Let for a given point P of C the point with the same abscissa be denoted by AP, the point with the same ordinate by BP, and let T denote the transformation

* Received May 9, 1940; Revised October 17, 1940.

[1] See J. Hadamard, "Équations aux dérivées partielles, le cas hyperbolique," *L'Enseignement Mathématique*, vol. 35 (1936), pp. 25-29.

[2] See the discussion in Courant-Hilbert, *Methoden der mathematischen Physik*, vol. II, p. 176.

[3] Following the notation of A. Huber, "Die erste Randwertaufgabe für geschlossene Bereiche bei der Gleichung $\partial^2 z/\partial x \partial y = f(x, y)$," *Monatshefte für Mathematik und Physik*, vol. 39 (1932), pp. 79-100.

BA, i. e. $TP = B(AP)$. The vertices of C are the fixed points of the transformations A and B. The sequence of points $P, TP, T^2P, T^3P, \cdots$ may be called the λ-polygon determined by P.[4] If there is an $n > 0$ for which $T^nP = P$, we call the smallest such n the " period " of P, and P a periodic point of C. In his paper A. Huber seeks to reduce the Dirichlet problem for general C to that for the case of a C with only two or three different vertices, and to that in which all points of C are of period 2. The method however does not seem to be applicable to all cases.[5]

Huber also treats a special case, in which the points of a λ-polygon are everywhere dense on C, namely the ellipse

$$x = a \cos t, \qquad y = b \sin (t - \xi), \tag{3}$$

where ξ/π is irrational. In this case the Dirichlet problem can be solved for general classes of boundary values, provided ξ satisfies certain conditions (which actually are of a Diophantine character).

In a recent paper D. G. Bourgin and R. Duffin give a complete discussion with the help of Fourier series of the case where C is a rectangle with sides of slope ± 1.[6] If ξ denotes the ratio of the sides of the rectangle, the solution of the Dirichlet problem is uniquely determined if and only if ξ is irrational; the solution exists for all boundary values, which are differentiable a sufficient number of times, if ξ cannot be approximated " too rapidly " by rationals.

The present paper takes up the case of an *arbitrary* closed curve C, which is convex in the x- and y-direction. The main feature of the theory developed here is the close connection established between the Dirichlet problem for C and the topological properties of the transformation T of C.

It is known that the Dirichlet problem for the potential equation (2) for a curve C is closely associated with the theory of conformal mappings. As (2) is invariant under conformal mappings, the Dirichlet problem for C can be reduced to that for a circle by a conformal mapping of C on a circle; the existence of such a mapping for suitable curves is guaranteed by Riemann's mapping theorem; for a circle again the problem can be solved explicitly either with the help of Fourier series or by Poisson's formula. In a similar manner

---

[4] See Huber, *loc. cit.*, p. 94.

[5] See the remarks by Hadamard, *loc. cit.*

[6] " The Dirichlet problem for the vibrating string equation," *Bulletin of the American Mathematical Society*, vol. 45 (1939), pp. 851-859. The results are reformulated here for our form (1) of the equation, which can be transformed into the equation $u_{xx} - u_{tt} = 0$ by a simple rotation of axes. Also the Dirichlet problem for the ellipses (3) treated by Huber, can be transformed easily into that for rectangles with sides of slope ± 1

the Dirichlet problem for (1) is closely connected with the mappings leaving (1) invariant, namely the mappings of the form

$$x' = f(x), \qquad y' = g(y). \tag{4}$$

In order that two curves can be mapped on each other by a transformation (4) their respective T-transformations have to be topologically equivalent.

The main results of this paper may be summarized as follows: All Jordan curves C, which are convex in the x- and y- direction, belong to one of two classes:

A curve C of the first class may be divided into two arcs C_1 and C_2, such that the values of u on C_2 already completely determine those on C_1 uniquely. The set of boundary functions for which the Dirichlet problem has a solution is not even "dense."

The curves C of the second class are exactly those for which the λ-polygon of every point is everywhere dense on C. For every such curve there is a bi-continuous mapping of C by a transformation of the form (4) on a rectangle with sides of slope ± 1 and irrational ratio ξ of sides. In that case the Dirichlet problem is reduced to the one considered by Bourgin and Duffin. The ratio ξ is uniquely determined by C. ξ is an invariant of C under mappings of the form (4). Curves of the second class with the same ξ can be mapped on each other by a bi-continuous transformation of the form (4). For almost all values of ξ the Dirichlet problem will be solvable for all boundary functions which are differentiable a sufficient number of times. There are however special values of ξ, for which even analyticity of the boundary values does not guarantee existence of the solution.

The following sufficient condition for uniqueness of a solution will be proved: If the set of periodic points of C is at most denumerable, then the solution of the Dirichlet problem is uniquely determined.

1. Formulation of the Dirichlet problem. In what follows, we denote by C a closed, continuous, simple curve, which has with every parallel to the x- or y-axis at most two points in common.[7]

1.1. *If $u(x, y)$ is twice continuously differentiable in the open region B bounded by C and continuous in $B + C$, then $u = f(x) + g(y)$, where f and g are continuous in every point of $B + C$ with the possible exception of those*

[7] The last condition corresponds roughly to the restriction to simply connected regions in the case of the potential equation (2). The rôle of the interior of the region in potential theory is played here by the rectangle formed by the characteristic lines of support. (See Huber, *loc. cit.*, p. 80).

points of C, through which there are two characteristic lines of support of C (i. e. of multiple vertices of C).

Proof. If $(x_1, y) \subset B$ and $(x_2, y) \subset B$, then $(x, y) \subset B$ for all x with $x_1 \leqq x \leqq x_2$. Hence

$$u_y(x_2, y) - u_y(x_1, y) = \int_{x_1}^{x_2} u_{xy}(\xi, y)\,d\xi = 0;$$

thus $u_y(x, y) = \phi(y)$ for all $(x, y) \subset B$. Similarly $u_x(x, y) = \psi(x)$ for all $(x, y) \subset B$. If (x, y) and (x', y') are in B, then there is a broken line L in B joining those two points, which consists of a finite number of segments parallel to the x- or y-axis; then

$$u(x', y') - u(x, y) = \int_L u_x dx + u_y dy = \int_x^{x'} \psi(\xi)\,d\xi + \int_y^{y'} \phi(\eta)\,d\eta.$$

Consequently $u(x, y)$ is of the form $f(x) + g(y)$ in B, where $f(x)$ and $g(y)$ are continuous in B, as $\psi(x)$ and $\phi(y)$ are continuous. If the lines of support of C, which are parallel to the x- or y-axis, are given by

$$x = a, \qquad x = b, \qquad y = \alpha, \qquad y = \beta$$

respectively, then $f(x)$ is continuous for $a < x < b$, and $g(y)$ for $\alpha < y < \beta$. From the continuity of u in $B + C$ we can conclude easily that $f(x)$ and $g(y)$ are continuous as well in every simple vertex (i. e. one lying on only one characteristic line of support.)

In order to make possible a unified treatment of the various cases, we replace the Dirichlet problem by the following modified problem: Let there be given a function v on C. We say a function $u(x, y)$ is a solution of the Dirichlet problem for the boundary values v, if

(a) $$u(x, y) = f(x) + g(y),$$

where $f(x)$ is continuous for $a \leqq x \leqq b$ and $g(y)$ continuous for $\alpha \leqq y \leqq \beta$,

(b) $$u(x, y) = v \quad \text{for } (x, y) \text{ on } C.^{8}$$

---

[8] This appears reasonable in view of Theorem 1.1. Thus we drop the assumption of differentiability for u, which may be justified by using a suitable generalization to define u_{xy} for non-differentiable functions. This is essential for the purposes of this paper, only in so far, as it permits us to state theorems in a more simplified manner, without having to add the cumbersome conditions (mostly of a rather obvious kind), that insure differentiability. It is essential however for all results derived here, that u is assumed to be continuous. The continuity of f and g in the respective *closed* intervals, postulated here, is equivalent to continuity of u in $B + C$, with the possible exception of the cases where C has less than four vertices and one of the multiple vertices has the character of a cusp.

If for a function v on C the Dirichlet problem has a solution, we call v "regular." A function v on C is called "semi-regular," if there exists a sequence of *regular* functions v_n on C, such that

(a) $$\lim_{n\to\infty} v_n = v,$$

(b) $$\lim_{n\to\infty} (\text{total variation of } v - v_n \text{ on } C) = 0.$$[9]

Every regular, and hence also every semi-regular function v on C, is obviously continuous.

Let A, B, T denote the transformations of C into itself, defined in the introduction. Then

$$A^{-1} = A, \qquad B^{-1} = B, \qquad T = BA, \qquad AT = T^{-1}A, \qquad BT = T^{-1}B.$$

It is obvious that the Dirichlet problem for a function $v = v(P)$ on C consists in finding two functions $f(P)$ and $g(P)$, continuous on C, such that

$$v(P) = f(P) + g(P), \qquad f(AP) = f(P), \qquad g(BP) = g(P). \tag{5}$$

We have from (5)

$$g(P) - g(TP) = g(P) - g(BAP) = g(P) - g(AP) = v(P) - v(AP),$$
$$f(P) - f(TP) = v(P) - v(BP).$$

Hence, by induction, for every integer $n > 0$

$$\begin{aligned} g(P) - g(T^nP) &= \sum_{k=0}^{n-1} [v(T^kP) - v(AT^kP)] \\ f(P) - f(T^nP) &= \sum_{k=0}^{n-1} [v(T^kP) - v(BT^kP)]. \end{aligned} \tag{6}$$

2. Transformations of C into itself. We shall make use of certain known results concerning arbitrary topological transformations T of a Jordan curve C into itself.[10]

---

[9] The total variation of a function w on C is defined as the least upper bound of $\sum_{i=1}^{n-1} | w(P_i) - w(P_{i+1}) |$ for any n points $P_1, \cdots, P_n$ of C lying in the order indicated by the indices.

[10] The following papers may be consulted in this connection: H. Kneser, "Reguläre Kurvenscharen auf den Ringflächen," *Mathematische Annalen*, vol. 91 (1924), pp. 135-154; J. Nielsen, "Om topologiske Afbildninger af en Jordankurve paa sig selv," *Matematiske Tidsskrift B* (1928), pp. 39-46; A. Denjoy, "Courbes définies par les équations differentielles à la surface du tore," *Journal de Math. pures et appl.*, vol. 11 (9th series), 1932. pp. 333-375; E. R. van Kampen, "Topological transformations of a curve," *American Journal of Mathematics*, vol. 57 (1934), pp. 142-152.

Every ordered pair of different points P, Q of C determines uniquely an *open* set, the "arc" $I = (P, Q)$, consisting of those points R of C, for which P, R, Q determine the positive sense on C. We denote with TP the image point of P under the transformation T, with TI the set of imagepoints of the arc I. T may be called *even*, if it preserves sense, i. e. if $T(P, Q) = (TP, TQ)$; similarly T will be called *odd*, if always $T(P, Q) = (TQ, TP)$. P is called a "periodic" point of T, if there is an $n > 0$, such that $P = T^nP$; the smallest such n will be called the "period" of P.

If T is an even $1 \leftrightarrow 1$ continuous transformation of C into itself, obviously one and only one of the following cases presents itself:

I. All points of C are periodic (T is "periodic"),

II. C contains periodic and non-periodic points (T is "semi-periodic"),

III. No point of C is periodic; there is no P such that the set of points $P, TP, T^2P, \cdots$ is everywhere dense on C (T is "intransitive"),

IV. No point of C is periodic; for some point P the set of points P, $TP, T^2P, \cdots$ is everywhere dense on C (T is "transitive").

In case I is is easily shown, that all points have the *same* period n. If P is any point, the points $P, TP, \cdots, T^{n-1}P$ divide C into n non-overlapping arcs; if I is anyone of those arcs, the other arcs are given by $TI, T^2I, \cdots$, $T^{n-1}I$ respectively.

In case II all *periodic* points have the same period n. The set F of periodic points is closed. The complementary set $C - F$ consists of a denumerable number of arcs I with endpoints in F, each one of which arcs is fixed under T^n; we may call them "periodic" arcs. If I is a periodic arc, the arcs $I, TI, \cdots, T^{n-1}I$ are non-overlapping. If Q is a point of the periodic arc I, T^nQ is in I as well; let I_0 denote that arc with endpoints Q and T^nQ, which is contained in I; then all arcs T^kI_0 are non-overlapping; all arcs $T^{nk+m}I_0$ are contained in T^mI; $\lim_{k\to\infty} T^{nk+m}Q$ exists and is the same for all Q in I, and is either identical with T^mR or with T^mS, if $I = (R, S)$.[11] It is evident that, if T is represented by a regular analytic function, there can be only a finite number of periodic points, unless all points are periodic.

In case III let Q be any point of C, and let σ be the set of limit points of the set of points $Q, TQ, T^2Q, \cdots$. It is known, that σ is a nowhere dense perfect set, and is independent of the choice of Q.[12] If I is an arc of $C - \sigma$,

[11] See Denjoy, *loc. cit.*, pp. 340-341.

[12] See Nielsen, *loc. cit.*, p. 40-41; van Kampen, p. 145.

all T^kI are non-overlapping. If the transformation T is representable in the form $s' = f(s)$, using the length of arc s as parameter, and if $\frac{df}{ds}$ is continuous and of bounded variation, the case III cannot occur.[13] (Thus the transformation T of C defined in the introduction, cannot be intransitive, if the curvature of C varies continuously along the curve).

In case IV T is topologically equivalent to a rotation of a circle; i. e. there exists a real number ξ and a $1 \leftrightarrow 1$ continuous, sense-preserving mapping $t = f(P)$ of the points P of C on the points $e^{2\pi i t}$ of the unit circle in the complex plane, such that $f(TP) \equiv t + \xi \pmod 1$.[14] The constant ξ, which is necessarily irrational, is uniquely determined (mod 1) by T; ξ will be called the "modul" of T. The parameter t is uniquely determined on C (mod 1), except for an additive constant.

In what follows we denote again by A, B, T the particular transformations of C into itself defined in the introduction. We shall call the curve C periodic, semi-periodic, intransitive, or transitive respectively, if the transformation T determined by C has that character. We observe that a curve with less than four vertices is necessarily semi-periodic, as multiple vertices of C are fixed points of T and not every point is a fixed point of T. If C is transitive, the modul ξ, uniquely determined by T, will also be referred to as the "modul of C"; we also agree in this case to determine the arbitrary additive constant of the parameter t by the condition, that $t = 0$ is the parameter value of the lefthand vertex on C; we call the parameter t obtained in this way, the *canonical* parameter on C. The canonical parameter on a transitive curve C of modul ξ is characterized by the properties: a) the values of t (mod 1) are in $1 \leftrightarrow 1$ continuous correspondence with the points of C; b) the direction of increasing t corresponds to the positive sense on C; c) $t = 0$ corresponds to the left hand vertex on C; d) the transformation T is given by $t' \equiv t + \xi \pmod 1$.

2. 1. *If C is neither periodic nor transitive, there exists an arc I_0, such that none of the arcs T^kI_0 and AT^rI_0 has a point in common with any other one.*

Proof: T is either semi-periodic or intransitive. According to the previous discussion there exists in either case an arc I, such that no two arcs T^kI have a common point. Then for every point P there is at most one k such that T^kP is in I. As there are at most four vertices, I contains at most 4 points of the

---

[13] See Denjoy, *loc. cit.*, p. 372; van Kampen, *loc. cit.*, p. 149.

[14] Kneser, *loc. cit.*, pp. 141-144; Nielsen, *loc. cit.*, p. 39. This theorem is also related to the theorem by M. Morse and G. A. Hedlund ("Symbolic dynamics II," *American Journal of Mathematics*, vol. 57 (1940), pp. 17-19), that a Sturmian series is identical with a properly chosen mechanical sequence.

form T^kQ, where Q is a vertex of C. Let now I_0 be a subarc of I, not containing any one of these four points. No two arcs T^kI_0 will have common points. If moreover T^kI_0 and AT^rI_0 had a common point, then also I_0 and $T^{-k}AT^rI_0 = AT^{k+r}I_0$ would have points in common; as AT^{k+r} is an *odd* transformation, and as the arc I_0 has a common point with its image under AT^{k+r}, there would necessarily exist a point R in I_0, which is fixed under the transformation: $R = AT^{k+r}R$. If $k + r$ is even, say $= 2m$, this implies

$$R = AT^{2m}R, \qquad T^mR = AT^mR;$$

if $k + r$ is odd, say $= 2m - 1$, we have

$$R = AT^{2m-1}R, \qquad T^mR = TAT^mR = BT^mR;$$

in either case T^mR would be a vertex of C, contrary to the assumption that no point R of this kind lies in I_0. Thus no two arcs T^kI_0 and AT^rI_0 have a common point. Obviously no two arcs of the form AT^kI_0 and AT^rI_0 can have a common point either.

2. 2. *If C is periodic of period n, there exists an arc I_0 such that no two of the arcs $I_0, TI_0, \cdots, T^{n-1}I_0, AI_0, ATI_0, \cdots, AT^{n-1}I_0$ have a common point.*

Proof: According to the properties of T in case I, mentioned above, there is an arc I, such that $I, TI, \cdots, T^{n-1}I$ are without common points. There are at most four *different* points of the form T^kQ in I, where Q is a vertex. If I_0 is a subarc of I not containing any of those four points, I_0 satisfies the statement.

2. 3. *Let C be transitive and of modul ξ. Let t be the canonical parameter of a point P of C. Then the points AP and BP have canonical parameter values $-t$ and $\xi - t$ respectively. (It follows, that the vertices of C, as fixed points of A and B, have canonical parameters 0, $\frac{1}{2}$, $\frac{1}{2}\xi$, $\frac{1}{2}\xi + \frac{1}{2}$ respectively.)*

Proof: If t is the canonical parameter of the variable point P of C, the parameter of AP is a continuous function $\phi(t)$ (mod 1); as $AT = T^{-1}A$, we have the functional equation

$$\phi(t + \xi) = \phi(t) - \xi \pmod 1.$$

Hence, as $\phi(0) = 0$,

$$\phi(n\xi) = -n\xi \pmod 1$$

for all integers n. If s is an arbitrary real number, there is a sequence of integers n_k, such that $s = \lim_{k\to\infty} n_k\xi$ (mod 1), as ξ is irrational (Kronecker's theorem); it follows that $\phi(s) = -s$. As $B = TA$, we have $\xi - t$ as parameter of BP.

2.4. *If C_1 and C_2 are two transitive curves with the same modul ξ, there is a transformation*

$$x' = \phi(x), \qquad y' = \psi(y) \tag{7}$$

with monotone increasing continuous functions ϕ and ψ mapping C_1 on C_2.[15]

Proof: Let t denote the canonical parameter on both C_1 and C_2. Let $R = (x, y)$ be a point interior to C_1. Let (x_1, y), (x_2, y), (x, y_1), (x, y_2) be the four points of intersection of C with parallels to the x- and y-axes through R. Let $x_1 < x_2$, $y_1 < y_2$, and let t_1, t_2, t_3, t_4 be the canonical parameters of those four points. Then $t_2 \equiv \xi - t_1$, $-t_4 \equiv t_3$ (mod 1). Take the points P_1, P_2, P_3, P_4 with canonical parameters t_1, t_2, t_3, t_4 on C_2. P_1 and P_2 have the same ordinate, P_3 and P_4 the same abscissa; let the image R' of R be defined as the intersection of the lines P_1P_2 and P_3P_4. It is easily seen that that mapping has the required properties.

2.5. *Every transitive curve of modul ξ can be represented in the form*

$$x = \phi(\cos 2\pi(t + \xi)), \qquad y = \psi(\sin 2\pi t),$$

where ϕ and ψ are monotone increasing continuous functions.

Proof: $x' = \cos 2\pi(t + \xi)$, $y' = \sin 2\pi t$ is an ellipse with t as canonical parameter and of modul ξ. Our curve can be mapped on this ellipse by a transformation of the form (7).

2.6. *Every transitive curve of modul ξ can be mapped by a transformation (7) with continuously increasing functions $\phi(x)$ and $\psi(y)$ on a rectangle with sides of slope ± 1.*

Proof: Take a rectangle with sides of slope ± 1 and ratio of sides $\alpha = \xi/(1 - \xi)$. It is easily seen that if we take on the rectangle as parameter t the length of arc counted from the lefthand vertex and divided by the total circumference, then t is the canonical parameter. The rectangle is transitive and of modul ξ. Apply 2.4.[16]

The solution u of the Dirichlet problem for a transitive curve C will be

---

[15] The mapping can be shown to be uniquely determined. For related questions see G. Loehs, "Die Jordankurve im Kurvennetz," *Abh. Math. Seminars d. Hamburgischen Univ.*, vol. 9 (1933), pp. 134-146, and "Eine Randwertaufgabe für Sechseckgewebe," *ibid.*, pp. 260-264.

[16] A proof of the fact, that a rectangle with sides of slope ± 1 and ratio of sides a is periodic for rational a and transitive for irrational a, is given by Bourgin and Duffin, *loc. cit.*, p. 854. The statement is easily seen to be equivalent to the theorem on reflected rays in a square, given by König and Szüsz (see Hardy and Wright, *The Theory of Numbers*, p. 366 *et seq.*).

obtained in terms of the canonical parameter t. For a discussion of the question whether these solutions are differentiable with respect to x and y, one would have to see whether the length of arc s on C is a differentiable function of the canonical parameter t; s is of course a monotone continuous function of t. The points T^nP have the same order as the multiples of the modul $\xi(\text{mod } 1)$; these multiples are, in the average, distributed *uniformly* (mod 1)[17]; hence, if t_1 and t_2 are the canonical parameters of two points P and Q, $\Delta t = t_2 - t_1 = \lim\limits_{n\to\infty} \frac{\mu(n)}{n}$, where $\mu(n)$ is the number of positive integers $k < n$, for which T^kP is contained in the arc (P, Q); if Δs is the distance of P and Q, $\frac{ds}{dt} = \lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$. It seems to be an open question, how to guarantee the existence and continuity of $\frac{ds}{dt}$ by suitable conditions on the transformation T, i. e. by conditions on the curve C.

3. Uniqueness and necessary conditions for existence of solutions of the Dirichlet problem.

3. 1. *The solution of the Dirichlet problem is uniquely determined, unless C contains a non-denumerable number of periodic points of T.*

Proof: Let $v(P) \equiv 0$ on C. According to (6) $g(T^nP) = g(P)$ for all P and n. Thus, as g is continuous, $g(Q) = g(P)$ whenever the sets of points $P, TP, T^2P, \cdots$ and $Q, TQ, T^2Q, \cdots$ have limit-points in common. In case T is intransitive or transitive this is the case for any two points P, Q; hence $g = \text{const.}$ on C. If T is semiperiodic, the two sets have common points whenever P and Q lie on the same arc free of periodic points; thus in the semiperiodic case g is constant on every arc free of periodic points; if the set of periodic points is denumerable, it follows that $g = \text{const.}$ throughout C.[18] Then also $f(P) = v - g = \text{const.}$ Thus $u = f + g$ is constant inside C, and as $u = 0$ on C, it follows that the solution of the Dirichlet problem determined by v vanishes identically.

3. 2. *Unless C is transitive, there exists an arc I_0 on C, such that for every regular or semi-regular $v(P)$ the values of v on $C - I_0$ determine uniquely those on I_0.*[19]

---

[17] See H. Weyl, "Ueber Gleichverteilung von Zahlen mod Eins," *Mathematische Annalen*, vol. 77 (1916).

[18] See Hobson, *The Theory of Functions of a Real Variable*, 3rd ed., vol. I, p. 364.

[19] In particular this is the case (at least in our formulation of the Dirichlet problem), if C has only two or three vertices. See Hadamard, *loc. cit.*, p. 26.

Proof: α) Let C be periodic of period n. Then $T^nP = P$ for all P. Then for regular v from (6)

$$\sum_{k=0}^{n-1} [v(T^kP) - v(AT^kP)] = 0$$

for all P. The same relation follows for all semi-regular v. Let I_0 be the arc determined in 2.2. Then for $P \subset I_0$, $v(P)$ is uniquely determined by the values of v outside I_0.

β) Let C be semi-periodic or intransitive. Let I_0 be determined by 2.1. Let P and Q be points of I_0. As all arcs T^kI_0 are non-overlapping, it follows, that the distance of T^kP and T^kQ tends towards 0, as k tends towards $+\infty$ or $-\infty$. Hence

$$\lim_{n\to\infty} [g(T^nP) - g(T^nQ)] = \lim_{n\to-\infty} [g(T^nP) - g(T^nQ)] = 0.$$

Thus, applying (6) to P and Q and subtracting, we obtain

$$(8) \qquad \sum_{k=-\infty}^{+\infty} ([v_m(T^kP) - v_m(T^kQ)] - [v_m(AT^kP) - v_m(AT^kQ)]) = 0$$

for any regular function v_m. Let v be semi-regular and $v \equiv 0$ on $C - I_0$. Let v_m be a sequence of regular functions with $\lim_{m\to\infty} v_m = v$ and

$$\lim_{m\to\infty} (\text{total variation of } v - v_m \text{ on } C) = 0;$$

then also

$$\lim_{m\to\infty} (\text{total variation of } v_m \text{ on } C - I_0) = 0.$$

As all arcs T^kI_0 and AT^rI_0 are non-overlapping, we have from (8)

$$| v_m(P) - v_m(Q) | \leqq (\text{total variation of } v_m \text{ on } C - I_0)$$

Hence

$$| v(P) - v(Q) | = \lim_{m\to\infty} | v_m(P) - v_m(Q) | = 0$$

for P and Q in I_0. As $v \equiv 0$ outside I_0, it follows, that $v \equiv 0$ in I_0 as well.

3.3. *Let C be non-transitive. Let C be referred to any continuous parameter s, (say $0 \leqq s \leqq 1$). Then there are functions $v(P)$ represented by analytic functions of s (of period 1), which are not semi-regular.*

Proof: Let v be a continuously differentiable function of s, vanishing on $C - I_0$ and not identically 0 on I_0. There are analytic functions v_m of s of period 1, such that $\lim_{m\to\infty} v_m = v$, $\lim_{m\to\infty}$ (total variation of $v - v_m$) $= 0$. If every

analytic function of s of period 1 were semi-regular, v would be semi-regular as well, contrary to 3. 2.

We may say then, that unless C is transitive, C may be split up into two arcs, such that the values of a regular or semi-regular function on one arc already completely determine those on the other arc. The curves C, which are not transitive, form the first class of curves of the introduction.

4. The Dirichlet problem for a transitive curve.

Theorem 2. 6 reduces the Dirichlet problem for any curve of the second class, i. e. for any transitive curve C, to the corresponding problem for a rectangle, treated in the paper by Bourgin and Duffin. This section contains some additions to the observations made in the paper just quoted. They are formulated for an arbitrary transitive curve C of modul ξ and canonical parameter t. We have from (5) and 2. 3.: $v(t)$ is regular, if and only if there exist continuous functions $f(t)$ and $g(t)$ of period 1, such that

$$v(t)=f(t)+g(t), \qquad f(t)=f(-t), \qquad g(t)=g(\xi-t).$$

Let

4. 1. $$S_n=\sum_{k=0}^{n-1}[v(k\xi)-v(-k\xi)].$$

Necessary and sufficient for regularity of the continuous function v is, that for every $\epsilon>0$ there is a $\delta=\delta(\epsilon)$, such that whenever n and m are integers with $|n\xi-m|<\delta$, then $|S_{N+n}-S_N|<\epsilon$ for all N.

Proof: a) Necessity of the condition: According to (6) and 2. 3

$$g(t)-g(t+n\xi)=\sum_{k=0}^{n-1}[v(t+k\xi)-v(-t-k\xi)].$$

Apply this formula to $t=N\xi$, using uniform continuity of g.

b) Sufficiency: Let for $t=\lim_{k\to\infty} n_k\xi \pmod 1$ the value of $g(t)$ be defined by $g(t)=-\lim_{k\to\infty} S_{n_k}$. Then $g(t)$ is uniquely defined and continuous, as $|S_{n_i}-S_{n_k}|<\epsilon$ for $|n_i\xi-m_k\xi-m|<\delta(\epsilon)$. Moreover, as v is continuous

(9) $$g(t+\xi)-g(t)=-\lim_{k\to\infty}[S_{n_k+1}-S_{n_k}]=v(-t)-v(t);$$

hence, by induction over n, using $g(0)=0$,

$$g(-n\xi)=-\sum_{k=1}^{n}[v(k\xi)-v(-k\xi)]=-S_{n+1}.$$

Consequently $g(-t)=g(t+\xi)$, $g(t)=g(\xi-t)$. Putting

$$f(t) = v(t) - g(t),$$

it follows from (9), that

$$f(-t) = v(-t) - g(-t) = v(-t) - g(t+\xi) = v(t) - g(t) = f(t).$$

Hence $v(t)$ is regular.

4.2. *All functions $v(t) = e^{2\pi i N t}$ for integers N are regular.*

Proof: In that case

$$S_n = -i \sin 2\pi n N\xi + i(1 - \cos 2\pi n N\xi)\, tan(\pi N\xi),$$

and $| S_{m+n} - S_m |$ is small for $\eta\xi$ small (mod 1) [although not uniformly in N].

4.3. *All functions $v(t)$ with continuous derivative with respect to t are semi-regular.* (Compare with 3.3.)

Proof: For every such function v there exist functions $v_m(t)$ of the form $\sum_{N=-n}^{n} a_N e^{2\pi i N t}$, such that

$$\lim_{m\to\infty} v_m = v$$
$$\lim_{m\to\infty} (\text{total variation of } v_m - v) = 0.$$

4.4. *A necessary condition for regularity of $v(t)$ is the existence of a constant M such that*

$$\left| \int_0^1 v(t) \sin 2\pi m t\, dt \right| < M\epsilon,$$

whenever m and n are integers such that $| m\xi - n | < \epsilon$.

Proof: From (9)

$$\int_0^1 g(t+\xi) \sin 2\pi m t\, dt - \int_0^1 g(t) \sin 2\pi m t\, dt$$
$$= \int_0^1 v(-t) \sin 2\pi m t\, dt - \int_0^1 v(t) \sin 2\pi m t\, dt;$$

hence

$$-2\int_0^1 v(t) \sin 2\pi m t\, dt = (1 - \cos 2\pi m\xi) \int_0^1 g(t) \sin 2\pi m t\, dt$$
$$+ \sin(2\pi m\xi) \int_0^1 g(t) \cos 2\pi m t\, dt.$$

This implies 4.4.

Let e. g. the modul ξ of C be a transcendental number of Liouville, i. e. for every k there are p and q with

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^{k+1}}.$$

Then differentiability of v of any fixed finite order is not sufficient for regularity, as from 4. 4.

$$\left| \int_0^1 v(t) \sin 2\pi\, qt\, dt \right| < \frac{M}{q^k},$$

whereas

$$v(t) = \sum_{k=1}^{\infty} \frac{\sin 2\pi n t}{n^{k-1}}$$

is a function $(k-3)$-times continuously differentiable and not satisfying that condition.

There are values of ξ, for which even analyticity of v is not sufficient for regularity. Let e. g. n_k be defined by

$$n_0 = 1, \qquad n_{k+1} = 3^{n_k}.$$

Then for $\xi = \sum_{k=0}^{\infty} n_k^{-1}$

$$n_i \xi \equiv \sum_{k=i+1}^{\infty} n_k^{-1} \pmod 1$$

$$\left| \sum_{k=i+1}^{\infty} n_k^{-1} \right| < \frac{2}{n_{i+1}} = \frac{2}{3^{n_i}}$$

$$|\sin 2\pi n_i \xi| < \frac{2}{3^{n_i}}$$

$$\left| \int_0^1 v(t) \sin 2\pi n_i t\, dt \right| < \frac{M}{3^{n_i}}$$

whereas for the analytic function

$$v(t) = \sum_{n=0}^{\infty} \frac{\sin 2\pi\, nt}{2^n}$$

we obtain

$$\int_0^1 v(t) \sin 2\pi\, nt\, dt = \frac{1}{2^{n+1}}.$$

Sufficient conditions for regularity in terms of Diophantine properties of ξ are given by Bourgin and Duffin in the paper mentioned above.

University of Kentucky.

ON VOLUME INTEGRAL INVARIANTS OF NON-HOLONOMIC DYNAMICAL SYSTEMS.*

By Clair J. Blackall.

1. Introduction. We consider non-holonomic dynamical systems with n independent position coördinates $q_1, q_2, \cdots, q_n$, subject to the non-integrable constraints,

$$\text{(a)} \qquad \sum_{\alpha=1}^{n} a_{r\alpha}\dot{q}_\alpha = 0, \qquad (r = 1, 2, \cdots, k,\ k < n).$$

We assume that the matrix $(a_{r\alpha})^{r=1,\cdots,k}_{\alpha=1,\cdots,n}$ is of rank k; that the applied forces Q_α, $\alpha = 1, \cdots, n$, are functions of the q's only; that the kinetic energy $T = \frac{1}{2}\sum_{\alpha,\beta=1}^{n} t_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta$, where the t's are functions of the q's only, where $\dot{q}_\alpha = dq_\alpha/dt$, $\alpha = 1, \cdots, n$, and where the determinant of $(t_{\alpha\beta})^{\alpha=1,\cdots,n}_{\beta=1,\cdots,n}$ is of rank n and > 0. We assume also that $t_{\alpha\beta} = t_{\beta\alpha}$, $\alpha, \beta = 1, \cdots, n$. In doing this we lose no generality.

If we introduce the customary, admissible transformation $q_\alpha = q_\alpha$, $p_\alpha = \partial T/\partial\dot{q}_\alpha$, $\alpha = 1, \cdots, n$, we have $T = \frac{1}{2}\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}p_\alpha p_\beta$, where the g's are subject to the same remarks as the t's. The equations of motion of the system are [1]

$$(A_1) \qquad \begin{cases} dp_\alpha/dt = Q_\alpha - \partial T/\partial q_\alpha + \sum_{r=1}^{k} a_{r\alpha}\lambda_r, \\ dq_\alpha/dt = \partial T/\partial p_\alpha, \quad (\alpha = 1, \cdots, n), \end{cases}$$

where T is expressed in terms of the p's and q's, and where the λ's are Lagrange's multipliers. A manner of determining the λ's is indicated later.

In this discussion t represents the time. The state of the system for any t may be represented by a point in the $(p, q) = (p_1, \cdots, p_n, q_1, \cdots, q_n)$ phase space. The equations of motion define the motion of a particle in this space, i. e., if the initial position of such a particle is given we can follow its course.

We assume all the continuity necessary for subsequent manipulations.

2. Determination of the λ's.[2] Since

* Received January 23, 1939.

[1] Cfr. P. Appell, *Traité de mécanique rationnelle*, vol. 2 (1904), p. 396.

[2] D. C. Lewis, Jr., "On line integrals and differential equations, especially those of dynamics," *Bulletin of the American Mathematical Society* (April, 1927), p. 320.

$$dq_\alpha/dt = \partial T/\partial p_\alpha = \sum_{\beta=1}^{n} g_{\alpha\beta}p_\beta, \qquad (\alpha = 1, \cdots, n),$$

(a) may be written $\sum_{\alpha,\beta=1}^{n} a_{r\alpha}g_{\alpha\beta}p_\beta = 0$, $r = 1, \cdots, k$. Differentiation of these last equations with respect to t, combined with substitutions from (A_1), yields

$$\sum_{\alpha,\beta,\gamma=1}^{n} \frac{\partial(a_{r\alpha}g_{\alpha\beta})}{\partial q_\gamma}\frac{dq_\gamma}{dt}p_\beta + \sum_{\alpha,\beta=1}^{n}(a_{r\alpha}g_{\alpha\beta})\Big(Q_\beta - \frac{\partial T}{\partial q_\beta} + \sum_{r=1}^{k} a_{r\beta}\lambda_r\Big) = 0, \qquad (r = 1, \cdots, k).$$

Solving these equations for the λ's we have

$$\text{(b)} \qquad \lambda_r = \tfrac{1}{2}\sum_{\gamma,\delta=1}^{n} l_{r\delta\gamma}p_\delta p_\gamma - F_r(q), \qquad (r = 1, \cdots, k),$$

where the l's and F's have their necessary meanings and are functions of the q's only. It is worthy of note that the l's are independent of the applied forces $Q_1, Q_2, \cdots, Q_n$. We take $l_{r\delta\gamma} = l_{r\gamma\delta}$, $\gamma, \delta = 1, \cdots, n$, $r = 1, \cdots, k$. No generality is lost in doing this.

3. Equations (A_1) and equations (A_2).

$$\sum_{\alpha=1}^{n} a_{r\alpha}\frac{\partial T}{\partial p_\alpha} = c_r, \qquad (r = 1, \cdots, k),$$

where the c's are constants, are first integrals of (A_1). The *actual* motions of the system are confined to the $(2n - k)$-dimensional manifold defined by (a). If we use (a) to eliminate k of the p's, say $p_{n-k+1}, p_{n-k+2}, \cdots, p_n$, from (A_1) we obtain a system of $(2n - k)$-differential equations which represent the *actual* motions of the system and no others. We shall put the label (A_2) on these resulting equations. Equations (A_1) represent the motions represented by the equations (A_2) plus others which are not actually possible. Those not actually possible are the ones for which at least one of the c's is $\neq 0$.

4. Volume integral invariants.[3] The system of differential equations

$$\text{(B)} \qquad dx_i/dt = X_i(x_1, \cdots, x_n), \qquad (i = 1, \cdots, n),$$

defines the motion of a particle in the $(x_1, x_2, \cdots, x_n)$ space. If at time zero, an arbitrary n-dimensional region V_0 is selected, it may be considered to flow into a corresponding region V_t in time t.

---

[3] Hedrick and Dunkel's translation of Goursat's *A Course in Mathematical Analysis*, Part II of vol. II (1917), p. 85.

An integral $\int \cdots \int M(x_1, \cdots, x_n)dx_1 \cdots dx_n$ is said to be a volume integral invariant of (B) if $\int \underset{V_t}{\cdots} \int M\,dx_1 \cdots dx_n$ is independent of t.

A necessary and sufficient condition for an M which is continuous and has continuous first partial derivatives is that

$$\text{(C)} \qquad \sum_{i=1}^{n} \frac{\partial (MX_i)}{\partial x_i} = 0.$$

Some questions concerning (A_1) and (A_2) are now appropriate. Do such systems always have a non-trivial M? Do they never have a non-trivial M? Or is the answer somewhere between?

Some answers to questions of this sort are given in the subsequent discussion.

5. On volume integral invariants for (A_1).

THEOREM. *A necessary and sufficient condition for the existence of an invariant integral of the type* $\int \cdots \int M(q_1, \cdots, q_n)dp_1 \cdots dp_n\, dq_1 \cdots dq_n$ *for system* (A_1), *where* M *is different from zero and continuous with continuous first partial derivatives, is that* $\sum_{\alpha,\delta,\sigma=1}^{n} \sum_{r=1}^{k} a_{r\alpha} l_{r\delta\alpha} t_{\delta\sigma} dq_\sigma$ *be an exact differential* $[= -d \log M]$.

Proof. Assuming $M > 0$ (which causes no loss in generality) let us first write (C) of ¶ 4 in the form

$$\sum_{i=1}^{n} \frac{\partial M}{\partial x_i} X_i + M \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} = 0$$

which states that

$$\frac{d \log M}{dt} = -\sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i}.$$

For equations (A_1) this reads

$$\frac{d \log M}{dt} = -\sum_{\alpha=1}^{n} \sum_{r=1}^{k} a_{r\alpha} \frac{\partial \lambda_r}{\partial p_\alpha}.$$

From ¶ 2

$$\frac{\partial \lambda_r}{\partial p_\alpha} = \sum_{\delta=1}^{n} l_{r\delta\alpha} p_\delta, \qquad (r = 1, \cdots, k,\ \alpha = 1, \cdots, n).$$

From ¶ 1

$$p_\delta = \frac{\partial T}{\partial \dot{q}_\delta} = \sum_{\sigma=1}^{n} t_{\delta\sigma} \frac{dq_\sigma}{dt}, \qquad (\delta = 1, \cdots, n).$$

Hence we have

$$\frac{d \log M}{dt} = -\sum_{a,\delta,\sigma=1}^{n} \sum_{r=1}^{k} a_{ra} l_{r\delta a} t_{\delta\sigma} \frac{dq_\sigma}{dt}.$$

This equation may be written

$$\sum_{\sigma=1}^{n} \left(\frac{\partial \log M}{\partial q_\sigma} + \sum_{a,\delta=1}^{n} \sum_{r=1}^{k} a_{ra} l_{r\delta a} t_{\delta\sigma} \right) \dot{q}_\sigma \equiv 0.$$

which, due to the arbitrariness of the $\dot{q}$'s, requires that

$$\frac{\partial \log M}{\partial q_\sigma} = -\sum_{a,\delta=1}^{n} \sum_{r=1}^{k} a_{ra} l_{r\delta a} t_{\delta\sigma}, \qquad (\sigma = 1, \cdots, n)$$

and so

$$\sum_{a,\delta,\sigma=1}^{n} \sum_{r=1}^{k} a_{ra} l_{r\delta a} t_{\delta\sigma} dq_\sigma = -d \log M$$

which proves the theorem.

It is an interesting fact that this condition is independent of the applied forces.

6. A useful lemma.[4] The following is a useful lemma.

Hypotheses. The functions

$$\begin{aligned} &X_i(x_1, \cdots, x_n), && (i = 1, \cdots, n) \\ &f_s(x_1, \cdots, x_n), && (s = 1, \cdots, k,\ k < n) \\ &M(x_1, \cdots, x_n) > 0 && \end{aligned}$$

are continuous, have continuous partial derivatives of first and second order, and satisfy the following conditions:

(1) $\sum_{i=1}^{n} \frac{\partial (MX_i)}{\partial x_i} = 0$ whenever all the $f_s = 0$, $s = 1, \cdots, k$.

(2) $\sum_{i=1}^{n} \frac{\partial f_{s'}}{\partial x_i} X_i = 0$ whenever all the $f_s = 0$, $s', s = 1, \cdots, k$.

(2a) $\frac{\partial}{\partial x_\beta} \left(\sum_{i=1}^{n} \frac{\partial f_{s'}}{\partial x_i} X_i \right) = 0$ whenever all the $f_{s'} = 0$, β, $s' = 1, \cdots, k$.

(3) the rank of $\left\| \frac{\partial f_s}{\partial x_i} \right\|_{\substack{s=1,\ldots,k \\ i=1,\ldots\ldots,n}}$ is k in a region R to which we confine our attention.

[4] The writer's knowledge of this lemma came from a letter written to him by Dr. D. C. Lewis, Jr., of Cornell University. Cfr. Appell, *Mécanique rationnelle*, vol. 2 (1904), p. 444.

Conclusions I. A solution of the differential equations

$$\frac{dx_i}{dt} = X_i, \qquad (i = 1, \cdots, n), \tag{4}$$

which satisfies $f_s = 0$, $s = 1, \cdots, k$, initially, must always satisfy this condition.

II. The flow on the manifold $f_s = 0$, $s = 1, \cdots, k$, admits in R a volume integral invariant, whose integrand never vanishes and is continuous with continuous first partial derivatives.

Proof. Choose any $n - k$ functions $f_{k+1}(x_1, \cdots, x_n)$, $f_{k+2}(x_1, \cdots, x_n)$, $\cdots, f_n(x_1, \cdots, x_n)$ such that

$$\frac{\partial(f_1, f_2, \cdots, f_k, f_{k+1}, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$$

does not vanish near an arbitrary point P of the manifold $f_s = 0$, $s = 1, \cdots, k$. The equations

$$u_i = f_i(x_1, \cdots, x_n), \qquad (i = 1, \cdots, n), \tag{5}$$

then determine an admissible transformation with inverse

$$x_i = q_i(u_1, \cdots, u_n), \qquad (i = 1, \cdots, n). \tag{6}$$

valid in an n-dimensional neighborhood of P.[5]

If we transform system (4) with the help of these equations, we can write our differential system as follows:

$$\frac{du_i}{dt} = U_i(u_1, u_2, \cdots, u_k, u_{k+1}, \cdots, u_n), \qquad (i = 1, \cdots, n), \tag{7}$$

where

$$U_s = \sum_{a=1}^{n} \frac{\partial f_s}{\partial x_a} X_a, \qquad (s = 1, \cdots, k),$$

which in accordance with (2) vanishes whenever $u_1 = u_2 = \cdots = u_k = 0$. Hence any solution of

$$\begin{aligned} \frac{du_i}{dt} &= W_i(u_{k+1}, \cdots, u_n) \\ &= U_i(0, \cdots, 0, u_{k+1}, \cdots, u_n), \qquad (i = k + 1, \cdots, n), \end{aligned} \tag{8}$$

---

[5] Remarks. Condition (1) is fulfilled if (4) has a volume integral invariant. Conditions (2) and (2a) are fulfilled if the f's are first integrals of (4).

when joined to $u_1 = u_2 = \cdots = u_k = 0$ constitutes a solution of (7) lying on the manifold $f_s = u_s = 0$, $s = 1, \cdots, k$. The existence and uniqueness theorems for systems of differential equations [6] show that these are all the solutions of (7) or (4) lying on this manifold initially. Thus I has now been proved (on the basis of hypotheses (2) and (3) only).

Let V be any n-dimensional region in the neighborhood of P.

Let V_t be the n-dimensional region into which the region V is carried after a time t according to the law of flow defined by the differential equations (4) or (7).

Let $J(t) = \int \underset{V_t}{\cdots} \int M\, dx_1 \cdots, dx_n$. Then

$$J'(0) = \int \underset{V}{\cdots} \int \left(\sum_{i=1}^{n} \frac{\partial (MX_i)}{\partial x_i} \right) dx_1 \cdots dx_n \tag{9}$$

Let

$$N = M \left| \frac{\partial (x_1, \cdots, x_n)}{\partial (u_1, \cdots, u_n)} \right| = N(u_1, \cdots, u_n) > 0.$$

Then by the laws for transformation of multiple integrals

$$J(t) = \int \underset{V'_t}{\cdots} \int N\, du_1 \cdots du_n$$

where V'_t is the map of V_t. Hence

$$J'(0) = \int \underset{V'}{\cdots} \int \left(\sum_{i=1}^{n} \frac{\partial (NU_i)}{\partial u_i} \right) du_1 \cdots du_n. \tag{10}$$

Comparing (9) and (10) and taking into account the arbitrariness of V we can readily show that

$$\sum_{i=1}^{n} \frac{\partial (MX_i)}{\partial x_i} = \left[\sum_{i=1}^{n} \frac{\partial (NU_i)}{\partial u_i} \right] \frac{\partial (u_1, \cdots, u_n)}{\partial (x_1, \cdots, x_n)}. \tag{11}$$

Now, by hypothesis, the left hand side of (11) vanishes on the manifold $f_s = 0$, $s = 1, \cdots, k$. Hence the right hand side also vanishes on this manifold; i. e., when $u_1 = u_2 = u_3 = \cdots = u_k = 0$. So, on this manifold,

$$\sum_{i=1}^{n} \frac{\partial (NU_i)}{\partial u_i} = 0. \tag{12}$$

From hypothesis (2) we know that $U_s = 0$ on the manifold $f_s = 0$, $s = 1, \cdots, k$. From hypothesis (2a) we know that $\partial U_s / \partial u_s = 0$ on this manifold. These equations hold for $s = 1, \cdots, k$.

---

[6] George D. Birkhoff, "Dynamical systems," *American Mathematical Society Colloquium Publications*, vol. 9 (1927).

Hence, on the manifold, we have

$$\sum_{i=k+1}^{n} \frac{\partial(NU_i)}{\partial u_i} = 0.$$

Hence the system (8) admits the integral invariant

$$\int \cdots \int N(0, \cdots, 0, u_{k+1}, \cdots, u_n) du_{k+1} \cdots du_n$$

at least in the neighborhood of P.

This chain of remarks suffices to establish the conclusions of the lemma, inasmuch as it is certainly possible to embed the manifold $f_s = 0, s = 1, \cdots, k$ in a set of neighborhoods of points like P.

This lemma enables us to deduce the existence of a volume integral invariant for (A_2) when the existence has been demonstrated for (A_1). It also enables us to deduce the non-existence of a volume integral invariant for (A_1) when the non-existence has been demonstrated for (A_2).

7. On volume integral invariants for (A_2).

THEOREM. *Not all systems (A_2) have an $M \neq 0$ and analytic in the neighborhood of a selected equilibrium point, i. e., a point for which all the right members of (A_2) vanish.*

Proof. One example will prove this statement. Let us take $T = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 + \frac{1}{2}\dot{q}_3^2$ and let the applied forces $Q_\alpha = 0$, $\alpha = 1, 2, 3$. For (a) let us take $q_2 q_1 \dot{q}_1 - \dot{q}_3 = 0$. In this example the equations (A_1) are

$$\frac{dp_1}{dt} = \lambda_1 q_2 q_1, \quad \frac{dp_2}{dt} = 0, \quad \frac{dp_3}{dt} = -\lambda_1, \quad \frac{dq_1}{dt} = p_1, \quad \frac{dq_2}{dt} = p_2, \quad \frac{dq_3}{dt} = p_3,$$

where

$$\lambda_1 = \frac{-(q_2 p_1^2 + q_1 p_1 p_2)}{1 + q_1^2 q_2^2}.$$

(a) may be written $q_2 q_1 p_1 - p_3 = 0$. If we use this equation to eliminate p_3 from (A_1) we have the equations of motion (A_2) for this example. These are

$$\frac{dp_1}{dt} = -\frac{q_2^2 q_1 p_1^2 + q_1^2 q_2 p_1 p_2}{1 + q_1^2 q_2^2}, \quad \frac{dp_2}{dt} = 0, \quad \frac{dq_1}{dt} = p_1, \quad \frac{dq_2}{dt} = p_2, \quad \frac{dq_3}{dt} = q_2 q_1 p_1.$$

The origin $(0, 0, 0, 0, 0)$ will serve as the equilibrium point.

Let us assume the theorem false. Consider $M > 0$ (which is no loss in

generality) and let $W = \log M$. Expanding W in ascending powers of p_1 and p_2 we have

$$W = W_0 + W_1 p_1 + W_2 p_2 + \cdots$$

where the W's with subscripts are functions of the q's only. W satisfies the equation $dW/dt + \theta = 0$ which is

$$\text{(d)} \qquad \frac{\partial W}{\partial p_1} \lambda_1 q_1 q_2 + \frac{\partial W}{\partial p_2} \cdot 0 + \frac{\partial W}{\partial q_1} p_1 + \frac{\partial W}{\partial q_2} p_2 + \frac{\partial W}{\partial q_3} q_2 q_1 p_1 + \theta \equiv 0$$

where

$$\theta = \frac{\partial}{\partial p_1}\left(\frac{dp_1}{dt}\right) + \frac{\partial}{\partial p_2}\left(\frac{dp_2}{dt}\right) + \frac{\partial}{\partial q_1}\left(\frac{dq_1}{dt}\right) + \frac{\partial}{\partial q_2}\left(\frac{dq_2}{dt}\right) + \frac{\partial}{\partial q_3}\left(\frac{dq_3}{dt}\right)$$
$$= -\frac{2q_2^2 q_1 p_1 + q_1^2 q_2 p_2}{1 + q_1^2 q_2^2}.$$

Equation (d) may be written

$$\left[\frac{\partial W_0}{\partial q_1} + q_2 q_1 \frac{\partial W_0}{\partial q_3} - \frac{2q_2^2 q_1}{1 + q_1^2 q_2^2}\right] p_1 + \left[\frac{\partial W_0}{\partial q_2} - \frac{q_1^2 q_2}{1 + q_1^2 q_2^2}\right] p_2$$

+ terms of higher degree in the p's $\equiv 0$.

This, then, requires that

$$\begin{cases} \dfrac{\partial W_0}{\partial q_1} + q_2 q_1 \dfrac{\partial W_0}{\partial q_3} - \dfrac{2q_2^2 q_1}{1 + q_1^2 q_2^2} \equiv 0 \\ \dfrac{\partial W_0}{\partial q_2} - \dfrac{q_1^2 q_2}{1 + q_1^2 q_2^2} \equiv 0 \end{cases}$$

in the neighborhood of $(0, 0, 0, 0, 0)$.

From the second of these equations we see that $W_0 = \frac{1}{2} \log (1 + q_1^2 q_2^2) + \Omega(q_1, q_3)$ where Ω is arbitrary. Substituting in the first equation we have

$$\frac{-q_1 q_2^2}{1 + q_1^2 q_2^2} + \frac{\partial \Omega}{\partial q_1} + q_1 q_2 \frac{\partial \Omega}{\partial q_3} \equiv 0$$

which requires that

$$-q_1 q_2^2 + \frac{\partial \Omega}{\partial q_1} + q_1^2 q_2^2 \frac{\partial \Omega}{\partial q_1} + q_1 q_2 \frac{{}^{8}b\varrho}{\partial \Omega} + q_1^3 q_2^3 \frac{\partial \Omega}{\partial q_3} \equiv 0.$$

Since the coefficients of the powers of q_2 must vanish, it follows that $\partial\Omega/\partial q_1 = \partial\Omega/\partial q_3 = 0$ and hence $-q_1 q_2^2 = 0$ in the neighborhood of $(0, 0, 0, 0, 0)$. Since this is not possible, the theorem is proved.

8. On volume integral invariants for (A_1).

THEOREM. *Not all systems* (A_1) *have an* $M \neq 0$ *and analytic in the neighborhood of an equilibrium point.*

Proof. The theorem of ¶ 7 and the lemma of ¶ 6 suffice to prove this theorem.

9. On equations (A_1) having non-trivial M's. If the λ's are independent of the p's the equations (A_1) admit $\int \cdots \int dp_1 \cdots dp_n\, dq_1 \cdots dq_n$ as a volume integral invariant. The system afforded by a sphere rolling without sliding on a plane is a system of this type.

The following is an example of a system having a non-trivial M. Let $T = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 + \frac{1}{2}\dot{q}_3^2$, $Q_1 = Q_2 = 0$, $Q_3 = +\sin q_3$ and (a) be $\sin q_3 \dot{q}_1 - \dot{q}_2 = 0$. In this example

$$\lambda_1 = \frac{-\cos q_3 p_1 p_3}{1 + \sin^2 q_3}$$

and the equations (A_1) are

$$\frac{dp_1}{dt} = \frac{-\sin q_3 \cos q_3}{1 + \sin^2 q_3} p_1 p_3, \quad \frac{dp_2}{dt} = \frac{\cos q_3 p_1 p_3}{1 + \sin^2 q_3}, \quad \frac{dp_3}{dt} = + \sin q_3$$

$$\frac{dq_1}{dt} = p_1, \quad \frac{dq_2}{dt} = p_2, \quad \frac{dq_3}{dt} = p_3. \qquad \int \cdots \int \sqrt{1 + \sin^2 q_3}\; dp_1 dp_2 dp_3\, dq_1 dq_2 dq_3$$

is a volume integral invariant for this example of equations (A_1).

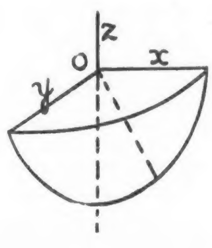

Fig. 1.

The origin O is taken at the center of the hollow sphere. The rectangular x, y, and z axes are fixed in space and taken as indicated. The position of the center of the rolling sphere is given by the spherical coördinates $(A - a, \alpha, \beta)$ where $A - a$ is the constant radius vector, α is the angle formed by the z axis and the radius vector, and β is the angle formed by the xz plane and the plane determined by the z axis and the radius vector.

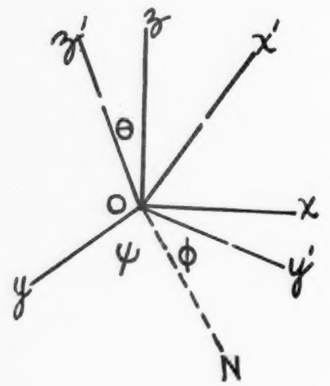

Fig. 2.

The orientation of the rolling sphere about its center is given by Eulerian angles taken as indicated in Fig. 2. The rectangular x, y, and z axes are, as stated above, fixed in space, and the rectangular x', y', and z' axes are fixed in the rolling sphere with origin O' at the center of the rolling sphere. By translation and no rotation we put the origin O' at O. We now have Fig. 2. N is the line of nodes in which the $x'y'$ plane of Fig. 2 cuts the xy plane. ψ, ϕ, θ are our Eulerian angles.

10. Another example: We consider here the case of a homogeneous sphere of radius (a) rolling without sliding in a hollow sphere of inner radius, (A).

Let
$$\Omega_x = \dot\theta \sin\psi - \dot\phi \sin\theta \cos\psi$$
$$\Omega_y = \dot\theta \cos\psi + \dot\phi \sin\theta \sin\psi$$
$$\Omega_z = \dot\psi + \dot\phi \cos\theta.$$

For the constraint equations we notice that:

I. $A\dot\alpha = -a\Omega_x \sin\beta + a\Omega_y \cos\beta$ and

II. $A\dot\beta \sin\alpha = -a\Omega_x \cos\beta \cos\alpha - \Omega_y \sin\beta \cos\alpha + \Omega_z \sin\alpha.$

These equations may be put in the form:

I. $A\dot\alpha = a\dot\theta \cos(\beta + \psi) + a\dot\phi \sin\theta \sin(\beta + \psi)$

II. $A\dot\beta \sin\alpha = -a\dot\theta \sin(\beta + \psi) \cos\alpha$
$+ a\dot\phi[\cos(\beta + \psi) \sin\theta \cos\alpha + \cos\theta \sin\alpha] + \dot\psi \sin\alpha.$

The kinetic energy

$$T=\frac{m}{2}(A-a)^2\dot{\alpha}^2+\frac{m}{2}(A-a)^2\dot{\beta}^2\sin^2\alpha+\tfrac{1}{2}I\dot{\theta}^2+\tfrac{1}{2}I\dot{\phi}^2+I\dot{\phi}\dot{\psi}\cos\theta+\tfrac{1}{2}I\dot{\psi}^2$$

where (m) is the mass of the rolling sphere and I is its moment of inertia with respect to a diameter.

Let us call $\alpha=q_1$, $\beta=q_2$, $\theta=q_3$. $\phi=q_4$, $\psi=q_5$. Then the constraint equations are:

$$\text{I.}\quad \frac{A}{a}\dot{q}_1-\dot{q}_3\cos(q_2+q_5)-\dot{q}_4\sin q_3\sin(q_2+q_5)=0$$

$$\text{II.}\quad \frac{A}{a}\dot{q}_2\sin q_1+\dot{q}_3\sin(q_2+q_5)\cos q_1 \\ -\dot{q}_4[\cos(q_2+q_5)\sin q_3\cos q_1+\cos q_3\sin q_1]-\dot{q}_5\sin q_1=0$$

$$H=\frac{m}{2}(A-a)^2\dot{q}_1^2+\frac{m}{2}(A-a)^2\dot{q}_2^2\sin^2 q_1+\tfrac{1}{2}I\dot{q}_3^2+\tfrac{1}{2}I\dot{q}_4^2 \\ +I\dot{q}_4\dot{q}_5\cos q_3+\tfrac{1}{2}I\dot{q}_5^2+U(q_1,q_2,q_3,q_4,q_5)$$

where U is the potential energy.

$$p_1=m(A-a)^2\dot{q}_1;\quad p_2=m(A-a)^2\dot{q}_2\sin^2 q_1;\quad p_3=I\dot{q}_3;$$
$$p_4=I\dot{q}_4+I\dot{q}_5\cos q_3;\quad p_5=I\dot{q}_4\cos q_3+I\dot{q}_5$$

so:

$$\dot{q}_1=\frac{p_1}{m(A-a)^2};\quad \dot{q}_2=\frac{p_2\csc^2 q_1}{m(A-a)^2};\quad \dot{q}_3=\frac{p_3}{I};$$
$$\dot{q}_4=\frac{p_4-p_5\cos q_3}{I\sin^2 q_3};\quad \dot{q}_5=\frac{-p_4\cos q_3+p_5}{I\sin^2 q_3}.$$

We may now write

$$H=\tfrac{1}{2}\frac{1}{m(A-a)^2}p_1^2+\tfrac{1}{2}\frac{\csc^2 q_1}{m(A-a)^2}p_2^2+\tfrac{1}{2}\cdot\frac{1}{I}p_3^2+\tfrac{1}{2}\frac{\csc^2 q_3}{I}p_4^2 \\ +\tfrac{1}{2}\frac{\csc^2 q_3}{I}p_5^2-\frac{\cos q_3\csc^2 q_3}{I}p_4p_5+U.$$

The equations of motion (A_1) are

$$\frac{dp_1}{dt}=\frac{\csc^2 q_1\cot q_1}{m(A-a)^2}p_2^2-\frac{\partial U}{\partial q_1}+\lambda_1\frac{A}{a};\frac{dp_2}{dt}=-\frac{\partial U}{\partial q_2}+\lambda_2\frac{A}{a}\sin q_1$$

$$\frac{dp_3}{dt}=\frac{\csc^2 q_3\cot q_3}{I}p_4^2+\frac{\csc^2 q_3\cot q_3}{I}p_5^2-\frac{\csc^3 q_3\,[1+\cos^2 q_3]}{I}p_4p_5 \\ -\frac{\partial U}{\partial q_3}-\lambda_1\cos(q_2+q_5)+\lambda_2\sin(q_2+q_5)\cos q_1$$

$$\frac{dp_4}{dt} = -\frac{\partial U}{\partial q_4} - \lambda_1 \sin q_3 \sin(q_2 + q_5) - \lambda_2[\cos(q_2 + q_5)\sin q_3 \cos q_1 + \cos q_3 \sin q_1]$$

$$\frac{dp_5}{dt} = -\frac{\partial U}{\partial q_5} - \lambda_2 \sin q_1$$

and the five equations for the ($\dot{q}$'s) in terms of the (p's) and (q's). The constraint equations may be written

$$\text{I.} \quad \frac{Ap_1}{am(A-a)^2} - \frac{\cos(q_2+q_5)}{I} p_3 - \frac{\csc q_3 \sin(q_2+q_5)}{I}(p_4 - p_5 \cos q_3) = 0$$

$$\text{II.} \quad \frac{A \csc q_1}{am(A-a)^2} p_2 + \frac{\sin(q_2+q_5)\cos q_1}{I} p_3 - \frac{\cos(q_2+q_5)\csc q_3 \cos q_1}{I} p_4$$
$$+ \frac{\cos(q_2+q_5)\csc q_3 \cos q_3 \cos q_1 - \sin q_1}{I} p_5 = 0.$$

Differentiation of I with respect to t and appropriate substitution yields

$$\frac{A}{am(A-a)^2}\left[\frac{\csc^2 q_1 \cot q_1}{m(A-a)^2} p_2^2 + \lambda_1 \frac{A}{a}\right] - \frac{\cos(q_2+q_5)}{I}\left[\frac{\csc^2 q_3 \cot q_3}{I} p_4^2\right.$$
$$+ \frac{\csc^2 q_3 \cot q_3}{I} p_5^2 - \frac{\csc^3 q_3[1+\cos^2 q_3]}{I} p_4 p_5 - \lambda_1 \cos(q_2+q_5)$$
$$\left. + \lambda_2 \sin(q_2+q_5)\cos q_1\right] - \frac{\csc q_3 \sin(q_2+q_5)}{I}$$
$$\cdot [-\lambda_1 \sin q_3 \sin(q_2+q_5) - \lambda_2\{\cos(q_2+q_5)\sin q_3 \cos q_1$$
$$+ \cos q_3 \sin q_1\} + \lambda_2 \cos q_3 \sin q_1] + \frac{\sin(q_2+q_5)}{I} p_3 \left[\frac{p_2 \csc^2 q_1}{m(A-a)^2}\right.$$
$$\left. + \frac{-p_4 \cos q_3 + p_5}{I \sin^2 q_3}\right] + \left[\frac{\csc q_3 \cot q_3 \sin(q_2+q_5)}{I^2} p_3 - \frac{\csc q_3 \cos(q_2+q_5)}{I}\right.$$
$$\left. \cdot \left\{\frac{p_2 \csc^2 q_1}{m(A-a)^2} + \frac{-p_4 \cos q_3 + p_5}{I \sin^2 q_3}\right\}\right][p_4 - p_5 \cos q_3] - \frac{\sin(q_2+q_5)}{I^2} p_3 p_5$$
$$- \text{a function } (q_1, q_2, q_3, q_4, q_5) = 0.$$

From this

$$\left[\frac{A^2}{a^2 m(A-a)^2} + \frac{1}{I}\right]\lambda_1 = -\frac{A \csc^2 q_1 \cot q_1}{am^2(A-a)^4} p_2^2 - \frac{\sin(q_2+q_5)\csc^2 q_1}{Im(A-a)^2} p_2 p_3$$
$$+ \frac{\csc q_3 \csc^2 q_1 \cos(q_2+q_5)}{Im(A-a)^2} p_2 p_4 - \frac{\csc q_3 \cos q_3 \csc^2 q_1 \cos(q_2+q_5)}{Im(A-a)^2} p_2 p_5$$
$$+ \text{a function } (q_1, q_2, q_3, q_4, q_5).$$

Differentiation of II with respect to t and appropriate substitution yields

$$\frac{A \csc q_1}{am(A-a)^2}\left[\lambda_2 \frac{A}{a} \sin q_1\right] + \frac{\sin(q_2+q_5)\cos q_1}{I}\left[\frac{\csc^2 q_3 \cot q_3}{I} p_4^2\right.$$

$$+ \frac{\csc^2 q_3 \cot q_3}{I} p_5^2 - \frac{\csc^3 q_3[1+\cos^2 q_3]}{I} p_4 p_5 - \lambda_1 \cos(q_2+q_5)$$

$$\left. + \lambda_2 \sin(q_2+q_5)\cos q_1\right] - \frac{\cos(q_2+q_5)\csc q_3 \cos q_1}{I}[-\lambda_1 \sin q_3 \sin(q_2+q_5)$$

$$- \lambda_2\{\cos(q_2+q_5)\sin q_3 \cos q_1 + \cos q_3 \sin q_1\}]$$

$$- \frac{\cos q_3 \csc q_3 \cos(q_2+q_5)\cos q_1 - \sin q_1}{I} \lambda_2 \sin q_1$$

$$- \frac{A \csc q_1 \cot q_1}{am^2(A-a)^4} p_1 p_2 - \frac{\sin(q_2+q_5)\sin q_1}{Im(A-a)^2} p_1 p_3$$

$$+ \frac{\cos(q_2+q_5)\cos q_1}{I} p_3 \left[\frac{p_2 \csc^2 q_1}{m(A-a)^2} + \frac{p_4 \cos q_3 + p_5}{I \sin^2 q_3}\right]$$

$$+ \frac{\sin(q_2+q_5)\csc q_3 \cos q_1}{I} p_4 \left[\frac{p_2 \csc^2 q_1}{m(A-a)^2} + \frac{-p_4 \cos q_3 + p_5}{I \sin^2 q_3}\right]$$

$$+ \frac{\cos(q_2+q_5)\csc q_3 \cot q_3 \cos q_1}{I^2} p_3 p_4 + \frac{\cos(q_2+q_5)\csc q_3 \sin q_1}{Im(A-a)^2} p_1 p_4$$

$$- \frac{\sin(q_2+q_5)\csc q_3 \cos q_3 \cos q_1}{I} \cdot p_5 \left[\frac{p_2 \csc^2 q_1}{m(A-a)^2}\right.$$

$$\left. + \frac{-p_4 \cos q_3 + p_5}{I \sin^2 q_3}\right] - \frac{\cos(q_2+q_5)\csc^2 q_3 \cos q_1}{I^2} p_3 p_5$$

$$- \frac{\cos(q_2+q_5)\cot q_3 \sin q_1}{Im(A-a)^2} p_1 p_5 - \frac{\cos q_1}{Im(A-a)^2} p_1 p_5$$

$$- \text{ a function of } (q_1, q_2, q_3, q_4, q_5) = 0.$$

From this

$$\left[\frac{A^2}{a^2 m(A-a)^2} + \frac{1}{I}\right]\lambda_2 = + \frac{A \csc q_1 \cot q_1}{am^2(A-a)^4} p_1 p_2 + \frac{\sin(q_2+q_5)\sin q_1}{Im(A-a)^2} p_1 p_3$$

$$- \frac{\cos(q_2+q_5)\csc q_3 \sin q_1}{Im(A-a)^2} p_1 p_4 + \frac{\cos(q_2+q_5)\cot q_3 \sin q_1 + \cos q_1}{Im(A-a)^2} p_1 p_5$$

$$- \frac{\cos(q_2+q_5)\cos q_1 \csc^2 q_1}{Im(A-a)^2} p_2 p_3 - \frac{\sin(q_2+q_5)\csc q_3 \cos q_1 \csc^2 q_1}{Im(A-a)^2} p_2 p_4$$

$$+ \frac{\sin(q_2+q_5)\csc q_3 \cos q_1 \csc^2 q_1}{Im(A-a)^2} p_2 p_5$$

$$+ \text{ a function of } (q_1, q_2, q_3, q_4, q_5).$$

The expression $\sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i}$ for this example is

$$\frac{1}{\left[\frac{A^2}{a^2m(A-a)^2}+\frac{1}{I}\right]}\left\{\frac{A}{a}\sin q_1\left[\frac{A\csc q_1\cot q_1}{am^2(A-a)^4}p_1\right.\right.$$

$$-\frac{\cos(q_2+q_5)\cos q_1\csc^2 q_1}{Im(A-a)^2}p_3-\frac{\sin(q_2+q_5)\csc q_3\cos q_1\csc^2 q_1}{Im(A-a)^2}p_4$$

$$\left.+\frac{\sin(q_2+q_5)\csc q_3\cos q_3\cos q_1\csc^2 q_1}{Im(A-a)^2}p_5\right]-\cos(q_2+q_5)$$

$$\cdot\left[-\frac{\sin(q_2+q_5)\csc^2 q_1}{Im(A-a)^2}p_2\right]+\sin(q_2+q_5)\cos q_1$$

$$\cdot\left[\frac{\sin(q_2+q_5)\sin q_1}{Im(A-a)^2}p_1-\frac{\cos(q_2+q_5)\cos q_1\csc^2 q_1}{Im(A-a)^2}p_2\right]$$

$$-\sin q_3\sin(q_2+q_5)\left[\frac{\csc q_3\csc^2 q_1\cos(q_2+q_5)}{Im(A-a)^2}p_2\right]$$

$$-\left[\cos(q_2+q_5)\sin q_3\cos q_1+\cos q_3\sin q_1\right]$$

$$\cdot\left[-\frac{\cos(q_2+q_5)\csc q_3\sin q_1}{Im(A-a)^2}p_1-\frac{\sin(q_2+q_5)\csc q_3\cos q_1\csc^2 q_1}{Im(A-a)^2}p_2\right]$$

$$-\sin q_1\left[\frac{\cos(q_2+q_5)\cot q_3\sin q_1+\cos q_1}{Im(A-a)^2}p_1\right.$$

$$\left.\left.+\frac{\sin(q_2+q_5)\cot q_3\cos q_1\csc^2 q_1}{Im(A-a)^2}p_2\right]\right\}.$$

This expression may be written

$$\frac{1}{\left[\frac{A^2}{a^2m(A-a)^2}+\frac{1}{I}\right]}\cdot\frac{A\cot q_1}{am(A-a)^2}\left[\frac{A}{am(A-a)^2}p_1-\frac{\cos(q_2+q_5)}{I}p_3\right.$$

$$\left.-\frac{\csc q_3\sin(q_2+q_5)}{I}(p_4-p_5\cos q_3)\right].$$

The quantity in brackets is exactly the left member of constraint equation I and, therefore, vanishes on the 8-dimensional manifold defined by I and II and on the 9-dimensional manifold defined by I. The reduced (A_2) system would, therefore, have a non-trivial volume integral invariant. This example is interesting because it illustrates the use of the lemma of paragraph 6.

COLLEGE OF ST. THOMAS,
ST. PAUL, MINNESOTA.

ON THE LAW OF THE ITERATED LOGARITHM.*

By Philip Hartman and Aurel Wintner.

1. Let $z_1(t), z_2(t), \cdots$ be an infinite sequence of real-valued independent functions[1] of class (L^2) on the interval $0 \leqq t \leqq 1$. Suppose that the expected value of $z_1(t) + \cdots + z_n(t)$ vanishes for every n, and that its standard deviation becomes infinite as $n \to \infty$; in other words, that

$$(1) \qquad \int_0^1 z_n(t)\,dt = 0, \qquad (n = 1, 2, \cdots),$$

and that, as $n \to \infty$,

$$(2) \qquad B_n \to \infty, \text{ where } B_n = b_1 + \cdots + b_n \text{ and } b_n = \int_0^1 [z_n(t)]^2 dt.$$

Kolmogoroff's law of the iterated logarithm[2] states that

$$(3) \qquad \overline{\lim_{n\to\infty}} \frac{z_1(t) + \cdots + z_n(t)}{(2B_n \log\log B_n)^{\frac{1}{2}}} = 1 \text{ for almost all } t,$$

provided that every $z_n(t)$ is a bounded function and its bound is subjected to the limitation

$$(4) \qquad \underset{0\leqq t\leqq 1}{\text{l. u. b.}} \, |z_n(t)| = o\left(\frac{B_n}{\log\log B_n}\right)^{\frac{1}{2}}, \text{ as } n \to \infty;$$

(it is understood that t-sets of measure zero may be neglected).

The literature[3] does not appear to contain a criterion which goes beyond this *boundedness* condition of Kolmogoroff; a condition which unfortunately is not satisfied in case of most of the sequences $\{z_n(t)\}$ which occur in standard applications. In order to see this, it is sufficient to consider the simplest possible case, that in which the distribution function of $z_n(t)$ is independent of n and has, in accordance with (1) and (2), a vanishing first moment and a finite non-vanishing second moment. Clearly, condition (4) then is satis-

* Received July 28, 1940.

[1] The independence of the functions or of "random variables" is meant in the sense in which it was always used in the analytic theory of probability; cf. e. g., A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin (1933), p. 150.

[2] A. Kolmogoroff, "Über das Gesetz des iterierten Logarithmus," *Mathematische Annalen*, vol. 101 (1929), pp. 126-135.

[3] Cf. P. Lévy, *Théorie de l'addition des variables aléatoires*, Paris (1937), pp. 258-289, where, in particular, Lévy's preceding investigation is presented ("La loi forte des grands nombres pour les variables aléatoires enchaînées," *Journal de Mathématique*, ser. 9, vol. 15 (1936), pp. 11-24, more particularly pp. 15-24).

fied only if the common distribution function of the $z_n(t)$ is constant outside a finite interval; so that the theorem breaks down even in the simplest cases to which the classical limit theorems of the theory of probability are applicable.

Nevertheless, one would expect that a law so fundamental as (3) is valid in this case of identical distributions, and even in the case of *nearly* identical distributions. But as far as the existing literature seems to go, it is essential to assume, not only that the distribution function of $z_n(t)$ be constant outside a finite interval (which may depend on n), but also that the length of this interval be

$$o\left(\frac{B_n}{\log\log B_n}\right)^{\frac{1}{2}}, \text{ as } n \to \infty. \tag{4 bis}$$

An example has even been constructed [4] to show this o-condition to be so essential, that the mere passage from o to O is capable of destroying the law of the iterated logarithm.

2. We shall, however, prove that the above conjecture as to the unrestricted validity of the law of the iterated logarithm in case of unbounded but equal, or *nearly* equal, distributions is nevertheless correct. In fact, the situation which occurs in the cases mentioned in § 1 is taken care of by the theorem to be proved, which may be formulated as follows:

Suppose that the first and second moments of the distribution functions

$$\sigma_1(x), \sigma_2(x), \cdots, \sigma_n(x), \cdots, \qquad (-\infty < x < +\infty),$$

of the independent functions

$$x_1(t), x_2(t), \cdots, x_n(t), \cdots, \qquad (0 \leqq t \leqq 1),$$

satisfy the conditions

$$\int_{-\infty}^{+\infty} x\, d\sigma_n(x) = 0 \tag{5}$$

and

$$\text{(6)}\quad \beta_n/n > \text{const.} > 0, \text{ where } \beta_n = \gamma_1 + \cdots + \gamma_n \text{ and } \gamma_n = \int_{-\infty}^{+\infty} x^2 d\sigma_n(x) < \infty,$$

and that the $\sigma_n(x)$ possess a dominant in the following sense: There exists a distribution function, $\tau(x)$, which has a second moment

[4] J. Marcinkiewicz and A. Zygmund, "Remarque sur la loi du logarithme itéré," *Fundamenta Mathematicae*, vol. 29 (1937), pp. 215-222.

$$\int_{-\infty}^{+\infty} x^2 d\tau(x) < \infty \tag{7}$$

and is such that

$$\int_{|x|\geqq r} d\sigma_n(x) = O\Big(\int_{|x|\geqq r} d\tau(x)\Big), \qquad r \to \infty, \tag{8}$$

holds uniformly in n. Then

$$\varlimsup_{n\to\infty} \frac{x_1(t) + \cdots + x_n(t)}{(2\beta_n \log\log \beta_n)^{\frac{1}{2}}} = 1 \text{ for almost all } t \tag{9}$$

(*and so, since* $x_n(t)$ *may be replaced by* $-x_n(t)$,

$$\varliminf_{n\to\infty} \frac{x_1(t) + \cdots + x_n(t)}{(2\beta_n \log\log \beta_n)^{\frac{1}{2}}} = -1 \text{ for almost all } t). \tag{9 bis}$$

The rôle of the assumptions (6) and (8) is to impose on the distributions σ_n a uniform lower and upper estimate, respectively (in fact, (6) is certainly satisfied if $0 < \text{const.} < \gamma_n < \infty$).

3. It is easy to see by a partial integration of

$$\int_{|x|\geqq r} f(x) d\{\sigma(x) - \rho(x)\},$$

that if $\sigma(x)$ and $\rho(x)$ are monotone non-decreasing functions for which

$$\int_{|x|\geqq r} d\sigma(x) \leqq \int_{|x|\geqq r} d\rho(x)$$

holds for *every* $r >$ const., and if $f(x)$ is any even, positive, continuous function which does not decrease when $|x|$ increases, then

$$\int_{|x|\geqq r} f(x) d\sigma(x) \leqq \int_{|x|\geqq r} f(x) d\rho(x)$$

($\leqq \infty$) also holds for every $r >$ const. On applying this remark to

$$\sigma = \sigma_n, \quad \rho = \tau; \qquad f(x) = |x|^\nu, \text{ where } \nu = 1, 2,$$

one sees that the assumptions (7) and (8) imply the estimates

$$\int_{|x|\geqq r} |x| d\sigma_n(x) = O\Big(\int_{|x|\geqq r} |x| d\tau(t)\Big), \qquad r \to \infty, \tag{10$_1$}$$

$$\int_{|x|\geqq r} x^2 d\sigma_n(x) = O\Big(\int_{|x|\geqq r} x^2 d\tau(t)\Big), \qquad r \to \infty, \tag{10$_2$}$$

uniformly for all n.

On the other hand, it is clear that there exists, for every distribution function τ which has a finite second moment, another distribution function, τ^*, which has a finite second moment and satisfies

$$\int_{|x|\geqq r} |x|\, d\tau(x) = o(\int_{|x|\geqq r} |x|\, d\tau^*(x)) \quad \text{and} \quad \int_{|x|\geqq r} x^2 d\tau(x) = o(\int_{|x|\geqq r} x^2 d\tau^*(x)),$$

as $r \to \infty$. Hence, on writing τ for τ^*, one sees that, without violating (7), the O of (10_1)–(10_2) may be replaced by o.

Accordingly, there exists a function $\phi = \phi(r)$, $0 < r < \infty$, which tends to 0 as $r \to \infty$ and is such that, for every n and r,

$$(10_1 \text{ bis}) \qquad \int_{|x|\geqq r} |x|\, d\sigma_n(x) \leqq \phi(r) \int_{|x|\geqq r} |x|\, d\tau(x).$$

Obviously, $\phi(r)$ may be chosen to be a decreasing function; so that

$$(11) \qquad 0 = \phi(+\infty) < \phi(r_1) < \phi(r_2) < \infty \quad \text{for} \quad \infty > r_1 > r_2 > 0.$$

Hence, one can construct a positive decreasing function $\epsilon = \epsilon(r)$, $0 < r < \infty$, which satisfies the conditions

$$(12_1) \quad \epsilon(r) > \phi(r^{1/9}); \qquad (12_2) \quad \epsilon(r) > r^{-1/6}(\log\log r)^{\frac{1}{2}}$$

and

$$(13) \qquad \epsilon(r) \to 0 \text{ as } r \to \infty,$$

and is such that the function

$$(14) \qquad \lambda(r) = \left(\frac{r}{\log\log r}\right)^{\frac{1}{2}} \epsilon(r) \text{ is monotone increasing,}$$

if const. $< r < \infty$. Then, by (12_2),

$$(15) \qquad \lambda(r) > r^{1/3}.$$

Clearly, one can assume that $\lambda(r)$ is defined for $0 < r \leqq$ const. in such a way that $\lambda(r)$ is monotone and satisfies (15) for $0 < r < \infty$.

4. In terms of the given sequence, $\{x_n(t)\}$, of independent functions, define on the same t-interval, $0 \leqq t \leqq 1$, another sequence, $\{z_n(t)\}$, of independent functions, by placing

$$(16) \qquad z_n(t) = \begin{cases} x_n(t) - \alpha_n, & \text{if } |x_n(t)| \leqq \lambda(n), \\ -\alpha_n, & \text{if } |x_n(t)| > \lambda(n), \end{cases}$$

where α_n denotes the number

$$(17) \qquad \alpha_n = \int_{|x|\leqq\lambda(n)} x\, d\sigma_n(x).$$

It will be shown that these functions $z_n(t)$ satisfy the three conditions (1), (2), (4) of Kolmogoroff for the validity of (3).

The relation (1) is clear from (16) and (17).

As to the second moment of the distribution function of $z_n(t)$, it is similarly seen that

$$\int_0^1 [z_n(t)]^2 dt = \int_{|x| \leqq \lambda(n)} x^2 d\sigma(x) - \alpha_n^2,$$

and so, according to (5) and the definition of γ_n in (6),

$$(18) \qquad \int_0^1 [z_n(t)]^2 dt = \gamma_n - \int_{|x| > \lambda(n)} x^2 d\sigma_n(x) - \Big(\int_{|x| > \lambda(n)} x d\sigma_n(x) \Big)^2.$$

But the Schwarz inequality and (10_2) imply that

$$(19_1) \quad \Big(\int_{|x| > \lambda(n)} x d\sigma_n(x) \Big)^2 \leqq \int_{|x| > \lambda(n)} x^2 d\sigma_n(x) = O\Big(\int_{|x| > \lambda(n)} x^2 d\tau(x) \Big), \quad (n \to \infty);$$

while (7) and (15) show that

$$(19_2) \qquad \int_{|x| > \lambda(n)} x^2 d\tau(x) \to 0, \qquad (n \to \infty).$$

Hence, from (18),

$$\int_0^1 [z_n(t)]^2 dt - \gamma_n \to 0.$$

Since this relation, when compared with (6), implies that

$$(20) \qquad B_n \sim \beta_n \text{ as } n \to \infty,$$

condition (2) is now verified.

Finally, it is clear from (17) that

$$\underset{0 \leqq t \leqq 1}{\text{l. u. b.}} \, | z_n(t) | \leqq \lambda(n) + \Big| \int_{|x| > \lambda(n)} x d\sigma_n(x) \Big| = O(\lambda(n)),$$

by (19_1)–(19_2). Hence, from (14) and (13),

$$\underset{0 \leqq t \leqq 1}{\text{l. u. b.}} \, | z_n(t) | = o\left(\frac{n}{\log \log n} \right)^{\frac{1}{2}}.$$

It follows, therefore, from (6) and (20) that (4) is satisfied.

This completes the proof of (3) for the functions (16).

5. In order to pass from (3) to (9), it will first be shown that if μ_n denotes the non-negative number

$$\mu_n = \int_{|x| > \lambda(n)} x d\sigma_n(x), \tag{21}$$

then

$$\sum_{k=16}^{\infty} \frac{\mu_k}{(k \log\log k)^{\frac{1}{2}}} < \infty; \tag{22}$$

(notice that $(\log\log k)^{\frac{1}{2}}$ is real (>0) only if $k \geqq 16$).

It is clear from (21) and (10_1 bis) that

$$\mu_k \leqq \phi(\lambda(k)) \int_{|x| > \lambda(k)} |x| \, d\tau(x).$$

Hence, if $L(n)$ is an abbreviation for the positive function

$$L(n) = (n \log\log n)^{-\frac{1}{2}}, \qquad (n \geqq 16), \tag{23}$$

the partial sum $\sum_{k=16}^{n}$ of (22) is majorized by the expression

$$\sum_{k=16}^{n} L(k)\phi(\lambda(k)) \int_{|x| > \lambda(k)} |x| \, d\tau(x),$$

which, after partial summation, appears in the form

$$\sum_{k=16}^{n-1} \sum_{j=16}^{k} L(j)\phi(\lambda(j)) \int_{\lambda(k) < |x| \leqq \lambda(k+1)} |x| \, d\tau(x) \tag{24}$$
$$+ \sum_{j=16}^{n} L(j)\phi(\lambda(j)) \int_{|x| > \lambda(n)} |x| \, d\tau(x).$$

Accordingly, the proof of (22) will be complete if one shows that the function (24) of n is $O(1)$, as $n \to \infty$.

To this end, notice first that, by (15) and (11),

$$\phi(\lambda(k^{1/3})) < \phi(k^{1/9}).$$

Hence, by (12_1),

$$\phi(\lambda(k^{1/3})) < \epsilon(k).$$

Since $\lambda(r)$ is monotone increasing [cf. (14)], it follows that

$$\sum_{k^{1/3} \leqq j < k} L(j)\phi(\lambda(j)) \leqq \phi(\lambda k^{1/3}) \sum_{k^{1/3} \leqq j < k} L(j) < \epsilon(k) \sum_{k^{1/3} \leqq j < k} L(j).$$

Consequently, by (23) and (14),

$$\sum_{k^{1/3} \leqq j < k} L(j)\phi(\lambda(j)) = \epsilon(k) O(k\, L(k)) = O(\lambda(k)).$$

On the other hand, (11) and (23) imply that

$$\sum_{16 \leqq j < k^{1/3}} L(j)\phi(\lambda(j)) = \sum_{16 \leqq j < k^{1/3}} L(j)O(1) = \sum_{16 \leqq j < k^{1/3}} O(1) = O(k^{1/3}).$$

Since (15) shows that $k^{1/3} = O(\lambda(k))$, it follows by addition of the last two relations that

$$\sum_{j=16}^{k} L(j)\phi(\lambda(j)) = O(\lambda(k)) + O(k^{1/3}) = O(\lambda(k)).$$

Hence, the function (24) of n is

$$\sum_{j=16}^{n-1} O(\lambda(k)) \int_{\lambda(k) < |x| \leqq \lambda(k+1)} |x| \, d\tau(x) + O(\lambda(n)) \int_{|x| > \lambda(n)} |x| \, d\tau(x)$$

$$= O\Big(\sum_{j=16}^{n-1} \int_{\lambda(k) < |x| \leqq \lambda(k+1)} x^2 d\tau(x)\Big) + O\Big(\lambda(n) \int_{|x| > \lambda(n)} |x| \, d\tau(x)\Big)$$

$$= O\Big(\int_{|x| \leqq \lambda(n)} x^2 d\tau(x)\Big) + O\Big(\int_{|x| > \lambda(n)} x^2 d\tau(x)\Big)$$

$$= O(1) + O(1), \text{ by } (7).$$

This proves (22).

6. In order to write (22) in the form in which it will be needed in the proof of (9), define for $0 \leqq t \leqq 1$ a function $y_n(t)$ as follows:

$$y_n(t) = \begin{cases} 0, \text{ if } |x_n(t)| \leqq \lambda(n), \\ x_n(t), \text{ if } |x_n(t)| > \lambda(n). \end{cases} \tag{25}$$

Then, since $\sigma_n(x)$ denotes the distribution function of $x_n(t)$,

$$\int_0^1 |y_n(t)| \, dt = \int_{|x| > \lambda(n)} |x| \, d\sigma_n(x). \tag{26}$$

Hence, from (21) and (22),

$$\sum_{k=16}^{\infty} \frac{\int_0^1 |y_k(t)| \, dt}{(k \log \log k)^{\frac{1}{2}}} < \infty. \tag{27}$$

It follows, therefore, from Fatou's inequality that

$$\sum_{k=16}^{\infty} \frac{|y_k(t)|}{(k \log \log k)^{\frac{1}{2}}} < \infty \text{ for almost all } t. \tag{28}$$

But a partial summation shows that if $\{a_k\}$ is any sequence for which the series

$$\sum_{k=16}^{\infty} \frac{a_k}{(k \log \log k)^{\frac{1}{2}}}$$

is convergent, then

$$a_1 + \cdots + a_n = o(n \log \log n)^{\frac{1}{2}}.$$

Hence, (28) and (27), respectively, imply that

$$(29) \qquad y_1(t) + \cdots + y_n(t) = o(n \log \log n)^{\frac{1}{2}} \text{ for almost all } t$$

and

$$(30) \qquad \int_0^1 |y_1(t)|\, dt + \cdots + \int_0^1 |y_n(t)|\, dt = o(n \log \log n)^{\frac{1}{2}}.$$

7. The proof of the theorem announced in § 2 is now immediate. In fact, it is clear from (16), (25) and (17), (26), respectively, that

$$|\alpha_n + z_n(t) - x_n(t)| \leqq |y_n(t)| \text{ and } |\alpha_n| \leqq \int_0^1 |y_n(t)|\, dt.$$

It follows, therefore, from (29) and (30) that

$$z_1(t) - x_1(t) + \cdots + z_n(t) - x_n(t) = o(n \log \log n)^{\frac{1}{2}} \text{ for almost all } t.$$

But $n \log \log n = O(\beta_n \log \log \beta_n)^{\frac{1}{2}}$, by (6). Consequently,

$$z_1(t) + \cdots + z_n(t) = x_1(t) + \cdots + x_n(t) + o(\beta_n \log \log \beta_n) \text{ for almost all } t.$$

Hence, (9) follows from (3) and (20).

QUEENS COLLEGE,
THE JOHNS HOPKINS UNIVERSITY.

A NEW DERIVATION OF THE EQUATIONS FOR THE DEFORMATION OF ELASTIC SHELLS.*

By Eric Reissner.

1. Introduction. The equations for the deformation of thin shells have first been established by A. E. H. Love [1] who thereby corrected and completed previous attempts by Aron [2] and Mathieu [3]. A reproduction of Love's work is found in his Treatise on the Mathematical Theory of Elasticity. The problem has been reexamined repeatedly. In this connection reference is made to the work of Krauss [4], Trefftz [5] and Odquist [6].

In the present paper an attempt is made to present the part of the theory concerned with small displacements in as simple a way as possible. In that respect two results in the following developments may be mentioned as significant. One is an elucidation of the reason why it is of special advantage to choose on the middle surface of the shell the lines of curvature as parametric curves. The other is a modified derivation of the stress-strain relations which utilizes directly the known expressions for the strain components with respect to orthogonal systems of coordinates and the assumption that the displacement components vary linearly with the distance along the normal from the middle surface of the shell.

2. Basic Assumptions of the Theory. The following assumptions are made:

1° The thickness of the shell is small compared with the radii of curvature of its middle surface.

2° The stress components normal to the middle surface are small compared with the other stress components and may be neglected in the stress-strain relations.

3° The normals of the undeformed middle surface are deformed into the normals of the deformed middle surface.

4° The displacements are so small that the equilibrium conditions for deformed elements are the same as if the elements were not deformed.

It should be said that 2° can be considered as a consequence of 1°, which

* Received August 11, 1940; Revised October 14, 1940. Presented to the American Mathematical Society, December 30, 1940.

is established by means of equilibrium considerations and that 3° follows in first approximation from 2° by means of the stress-strain relations.

The results of the theory are obtained in three main steps. First an appropriate system of coördinates on the shell is introduced and certain geometrical relations established. Then the equilibrium conditions for an element of the shell are formulated. Finally the system of equations is completed by deriving the relations between displacement components and stress resultant components.

3. The Coördinate System. The location of a point of the shell is given by three parameters, two of which vary on the middle surface of the shell and the third along the normal to the middle surface. The condition we impose on the parametric curves is that they form a *three*-dimensional orthogonal system. This condition is imposed because for a non-orthogonal system of coordinates the expressions for the strain components and the relations between stresses, stress resultants and displacements are considerably more complicated than in the case of orthogonal coordinates. In what follows vector calculus is employed which simplifies the presentation considerably.

The radius vector to a point of the shell may be written in the form

$$\boldsymbol{R}(\xi_1, \xi_2, \zeta) = \boldsymbol{r}(\xi_1, \xi_2) + \zeta \boldsymbol{n}(\xi_1, \xi_2) \tag{1}$$

where $\boldsymbol{r}$ denotes a vector to the middle surface; $\xi_1 = \text{const.}$ and $\xi_2 = \text{const.}$ are the parametric curves on the middle surface and ζ is the distance of the point from the middle surface measured along the unit normal vector $\boldsymbol{n}$.

We require that the line element has the form

$$ds^2 = A_1{}^2 d\xi_1{}^2 + A_2{}^2 d\xi_2{}^2 + C^2 d\zeta^2 \tag{2}$$

and find by a simple calculation that

$$ds^2 = d(\boldsymbol{r} + \zeta\boldsymbol{n}) \cdot d(\boldsymbol{r} + \zeta\boldsymbol{n}) \tag{3}$$

is of the form (2), provided

$$\frac{\partial \boldsymbol{r}}{\partial \xi_1} \cdot \frac{\partial \boldsymbol{r}}{\partial \xi_2} = 0, \quad \frac{\partial \boldsymbol{r}}{\partial \xi_1} \cdot \frac{\partial \boldsymbol{n}}{\partial \xi_2} + \frac{\partial \boldsymbol{r}}{\partial \xi_2} \cdot \frac{\partial \boldsymbol{n}}{\partial \xi_1} = 0. \tag{4}$$

It is well known that (4) is the characteristic property of the lines of curvature, for which also

$$\frac{\partial \boldsymbol{n}}{\partial \xi_m} = \frac{1}{R_m} \frac{\partial \boldsymbol{r}}{\partial \xi_m}, \qquad (m = 1, 2) \tag{5}$$

with R_m for the principal radii of curvature. With the notation

$$\frac{\partial \boldsymbol{r}}{\partial \xi_m} \cdot \frac{\partial \boldsymbol{r}}{\partial \xi_m} = \alpha_m{}^2, \qquad (m = 1, 2) \tag{6}$$

and with (4) and (5) there follows from (3)

$$ds^2 = \alpha_1^2\left(1+\frac{\zeta}{R_1}\right)^2 d\xi_1^2 + \alpha_2^2\left(1+\frac{\zeta}{R_2}\right)^2 d\xi_2^2 + d\zeta^2 \tag{7}$$

On setting

$$\boldsymbol{t}_m = \frac{1}{\alpha_m}\,\frac{\partial \boldsymbol{r}}{\partial \xi_m}, \qquad (m=1,2) \tag{8}$$

we have the following well-known formulae for the derivatives of the tangent unit vectors

$$\left\{\begin{aligned} \frac{\partial \boldsymbol{t}_m}{\partial \xi_m} &= -\frac{1}{\alpha_n}\,\frac{\partial \alpha_m}{\partial \xi_n}\,\boldsymbol{t}_n - \frac{\alpha_m}{R_m}\,\boldsymbol{n} \\ \frac{\partial \boldsymbol{t}_m}{\partial \xi_n} &= \frac{1}{\alpha_m}\,\frac{\partial \alpha_n}{\partial \xi_m}\,\boldsymbol{t}_n, \end{aligned}\right. \qquad \begin{aligned}(m&=1,2)\\(n&=2,1)\end{aligned} \tag{9[1]}$$

which are subsequently needed.

4. The Equilibrium Conditions. Consider an element of the shell bounded by surfaces $\xi = \text{const.}$, $\xi + d\xi = \text{const.}$ and $\zeta = \pm h/2$. Forces and moments acting on all six faces must be in equilibrium. We denote by $\boldsymbol{N}$ the force resultants and by $\boldsymbol{M}$ the moment resultants, per unit of length measured along the parametric curves on the middle surface. $\boldsymbol{N}_1$ and $\boldsymbol{M}_1$ act on the faces normal to $\boldsymbol{t}_1$ and $\boldsymbol{N}_2$ and $\boldsymbol{M}_2$ on the faces normal to $\boldsymbol{t}_2$. By $\boldsymbol{p}$ we denote the external force per unit of area of the middle surface. Taking into account that stress resultants as well as areas change with the coordinates of the middle surface the following two conditions of force and moment equilibrium must be satisfied:

$$\frac{\partial \alpha_2 \boldsymbol{N}_1}{\partial \xi_1} + \frac{\partial \alpha_1 \boldsymbol{N}_2}{\partial \xi_2} + \alpha_1\alpha_2\boldsymbol{p} = 0 \tag{10}$$

$$\frac{\partial \alpha_2 \boldsymbol{M}_1}{\partial \xi_1} + \frac{\partial \alpha_1 \boldsymbol{M}_2}{\partial \xi_2} + \alpha_2\boldsymbol{N}_1 \times \frac{\partial \boldsymbol{r}}{\partial \xi_1} + \alpha_1\boldsymbol{N}_2 \times \frac{\partial \boldsymbol{r}}{\partial \xi_2} = 0. \tag{11}$$

To obtain six scalar equations from (10) and (11), components of force and moment resultants with respect to normal and tangential directions are introduced as follows:

$$\boldsymbol{N}_1 = N_{11}\boldsymbol{t}_1 + N_{12}\boldsymbol{t}_2 + Q_1\boldsymbol{n} \tag{12}$$

$$\boldsymbol{N}_2 = N_{21}\boldsymbol{t}_1 + N_{22}\boldsymbol{t}_2 + Q_2\boldsymbol{n} \tag{13}$$

$$\boldsymbol{M}_1 = M_{11}\boldsymbol{t}_2 + M_{12}\boldsymbol{t}_1 \tag{14}$$

$$\boldsymbol{M}_2 = -M_{21}\boldsymbol{t}_2 + M_{22}\boldsymbol{t}_1. \tag{15}$$

---

[1] Eqs. (9) are obtained by writing the derivatives of the t_m's as combination with undetermined coefficients of n and the t_m's and determining the coefficients with the help of (5) and (6).

In these expressions moments are considered positive when they produce positive stresses on the part of the shell above the middle surface. They are represented as vectors such that the moments act in clockwise direction if one looks at the arrow head of the vectors. The absence of a third component in $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ is due to the fact that the width $\alpha d\xi$ of the faces is an infinitesimal while the height h is finite.

Introducing (12) to (15) into (10) and (11) we obtain

$$
\begin{aligned}
(16)\qquad & \left(\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{21}}{\partial \xi_2}\right)\boldsymbol{t}_1 + \left(\frac{\partial \alpha_2 N_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2}\right)\boldsymbol{t}_2 \\
& + \left(\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2}\right)\boldsymbol{n} + \alpha_2\left(N_{11}\frac{\partial \boldsymbol{t}_1}{\partial \xi_1} + N_{12}\frac{\partial \boldsymbol{t}_2}{\partial \xi_1} + Q_1\frac{\partial \boldsymbol{n}}{\partial \xi_1}\right) \\
& + \alpha_1\left(N_{21}\frac{\partial \boldsymbol{t}_1}{\partial \xi_2} + N_{22}\frac{\partial \boldsymbol{t}_2}{\partial \xi_2} + Q_2\frac{\partial \boldsymbol{n}}{\partial \xi_2}\right) + \alpha_1\alpha_2(p_1\boldsymbol{t}_1 + p_2\boldsymbol{t}_2 + q\boldsymbol{n}) = 0
\end{aligned}
$$

$$
\begin{aligned}
(17)\qquad & \left(\frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{22}}{\partial \xi_2}\right)\boldsymbol{t}_1 - \left(\frac{\partial \alpha_2 M_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{21}}{\partial \xi_2}\right)\boldsymbol{t}_2 \\
& + \alpha_2\left(M_{12}\frac{\partial \boldsymbol{t}_1}{\partial \xi_1} - M_{11}\frac{\partial \boldsymbol{t}_2}{\partial \xi_1}\right) + \alpha_1\left(M_{22}\frac{\partial \boldsymbol{t}_1}{\partial \xi_2} - M_{21}\frac{\partial \boldsymbol{t}_2}{\partial \xi_2}\right) \\
& + \alpha_1\alpha_2[(N_{11}\boldsymbol{t}_1 + N_{12}\boldsymbol{t}_2 + Q_1\boldsymbol{n}) \times \boldsymbol{t}_1 + (N_{21}\boldsymbol{t}_1 + N_{22}\boldsymbol{t}_2 + Q_2\boldsymbol{n}) \times \boldsymbol{t}_2] = 0.
\end{aligned}
$$

With (9), (5) and $\boldsymbol{t}_1 \times \boldsymbol{t}_2 = \boldsymbol{n}$ this becomes

$$
\begin{aligned}
(18)\qquad & \left(\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{21}}{\partial \xi_2} + N_{12}\frac{\partial \alpha_1}{\partial \xi_2} - N_{22}\frac{\partial \alpha_2}{\partial \xi_1} + Q_1\frac{\alpha_1\alpha_2}{R_1} + \alpha_1\alpha_2 p_1\right)\boldsymbol{t}_1 \\
& + \left(\frac{\partial \alpha_2 N_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2} + N_{21}\frac{\partial \alpha_2}{\partial \xi_1} - N_{11}\frac{\partial \alpha_1}{\partial \xi_2} + Q_2\frac{\alpha_1\alpha_2}{R_2} + \alpha_1\alpha_2 p_2\right)\boldsymbol{t}_2 \\
& + \left(\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2} - N_{11}\frac{\alpha_1\alpha_2}{R_1} - N_{22}\frac{\alpha_1\alpha_2}{R_2} + \alpha_1\alpha_2 q\right)\boldsymbol{n} = 0
\end{aligned}
$$

$$
\begin{aligned}
(19)\qquad & \left(\frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{22}}{\partial \xi_2} - M_{11}\frac{\partial \alpha_1}{\partial \xi_2} + M_{21}\frac{\partial \alpha_2}{\partial \xi_1} - Q_2\alpha_1\alpha_2\right)\boldsymbol{t}_1 \\
& - \left(\frac{\partial \alpha_2 M_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{21}}{\partial \xi_2} - M_{22}\frac{\partial \alpha_2}{\partial \xi_1} + M_{12}\frac{\partial \alpha_1}{\partial \xi_2} - Q_1\alpha_1\alpha_2\right)\boldsymbol{t}_2 \\
& - \alpha_1\alpha_2\left(\frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} + N_{12} - N_{21}\right)\boldsymbol{n} = 0.
\end{aligned}
$$

$\boldsymbol{t}_1$, $\boldsymbol{t}_2$ and $\boldsymbol{n}$ being linearly independent their coefficients in (18) and (19) have to vanish so that six scalar equations for the ten components of the stress resultants are obtained. The next step, expression of the stress resultants in terms of the stress components, will show however that only five of the six equations are relevant, since the coefficient of $\boldsymbol{n}$ in (19) vanishes identically.

To obtain stress resultants in terms of the stresses σ_1, σ_2 and τ_{12} observe, for instance, that by definition $\alpha_1 N_{11} = \int_{-h/2}^{h/2} \sigma_1 A_1 d\zeta$ where (2) and (7) contain the meaning of A_1. Dividing through by α_1 gives the desired expression for N_{11}. In the same way follow expressions for the other resultants. They are:

$$(20) \quad N_{11} = \int_{-h/2}^{h/2} \sigma_1 \left(1 + \frac{\zeta}{R_2}\right) d\zeta, \qquad M_{11} = \int_{-h/2}^{h/2} \sigma_1 \left(1 + \frac{\zeta}{R_2}\right) \zeta \, d\zeta$$

$$(21) \quad N_{12} = \int_{-h/2}^{h/2} \tau_{12} \left(1 + \frac{\zeta}{R_2}\right) d\zeta. \qquad M_{12} = \int_{-h/2}^{h/2} \tau_{12} \left(1 + \frac{\zeta}{R_2}\right) \zeta \, d\zeta$$

$$(22) \quad N_{21} = \int_{-h/2}^{h/2} \tau_{12} \left(1 + \frac{\zeta}{R_1}\right) d\zeta. \qquad M_{21} = \int_{-h/2}^{h/2} \tau_{12} \left(1 + \frac{\zeta}{R_1}\right) \zeta \, d\zeta$$

$$(23) \quad N_{22} = \int_{-h/2}^{h/2} \sigma_2 \left(1 + \frac{\zeta}{R_1}\right) d\zeta. \qquad M_{22} = \int_{-h/2}^{h/2} \sigma_2 \left(1 + \frac{\zeta}{R_1}\right) \zeta \, d\zeta.$$

In view of assumption 2° that the stress components normal to the middle surface are negligibly small. i. e. $\sigma_\zeta \approx 0$, the stress-strain relations for isotropic materials which involve the stress components occurring in (20) to (23) are:

$$(24) \quad \sigma_1 = \frac{E}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2), \quad \sigma_2 = \frac{E}{1-\nu^2}(\epsilon_2 + \nu\epsilon_1), \quad \tau_{12} = G\gamma_{12}.$$

To complete the system of equations the strain components have to be expressed in terms of the displacement components, taking into account assumption 3° concerning the deformation of the normal to the middle surface.

5. Determination of the Strain Components. We write the displacement vector in the form

$$(25) \qquad \boldsymbol{U} = U_1\boldsymbol{t}_1 + U_2\boldsymbol{t}_2 + W\boldsymbol{n}.$$

The strain components for a system of curvilinear coördinates corresponding to a line element of the form (2) are known to be

$$(26) \quad \epsilon_1 = \frac{1}{A_1}\frac{\partial U_1}{\partial \xi_1} + \frac{U_2}{A_1A_2}\frac{\partial A_1}{\partial \xi_2} + \frac{W}{A_1C}\frac{\partial A_1}{\partial \zeta}, \quad \epsilon_2 = \frac{1}{A_2}\frac{\partial U_2}{\partial \xi_2} + \frac{U_1}{A_1A_2}\frac{\partial A_2}{\partial \xi_1} + \frac{W}{A_2C}\frac{\partial A_2}{\partial \zeta}$$

$$(27) \qquad \gamma_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{U_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{U_1}{A_1}\right)$$

$$(28) \quad \gamma_{1\zeta} = \frac{C}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{W}{C}\right) + \frac{A_1}{C}\frac{\partial}{\partial \zeta}\left(\frac{U_1}{A_1}\right), \quad \gamma_{2\zeta} = \frac{C}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{W}{C}\right) + \frac{A_2}{C}\frac{\partial}{\partial \zeta}\left(\frac{U_2}{A_2}\right).$$

We proceed as follows to reduce (26) and (27), which give the com-

ponents occurring in the stress-strain relations, to their appropriate form. Since the normal to the undeformed middle surface is to remain straight we write for the displacement components

$$U = U(\xi_1, \xi_2, 0) + \zeta\left(\frac{\partial U(\xi_1, \xi_2, \zeta)}{\partial \zeta}\right)_{\zeta=0} = u + \zeta u' \tag{29}$$

$$W = W(\xi_1, \xi_2, 0) = w. \tag{30}$$

With the help of the condition that the angle between the middle surface and normal remains unchanged by the deformation we may express u'_1 and u'_2 in terms of u_1, u_2 and w. The changes of angle between the middle surface and normal being given by the strains $\gamma_{1\zeta}$ and $\gamma_{2\zeta}$ we have

$$(\gamma_{m\zeta})_{\zeta=0} = 0, \qquad (m = 1, 2). \tag{31}$$

According to (7)

$$C = 1, \qquad A_m = \alpha_m\left(1 + \frac{\zeta}{R_m}\right), \qquad (m = 1, 2).$$

so that with (29) and (30), (31) is equivalent to

$$u'_m = \frac{u_m}{R_m} - \frac{1}{\alpha_m}\frac{\partial w}{\partial \xi_m}, \qquad (m = 1, 2) \tag{32}$$

and thus u' is expressed by u and w. Substituting u'_1 and u'_2 from (32), the displacement components (29) take the following final form

$$U_m = u_m - \zeta\left(\frac{1}{\alpha_m}\frac{\partial w}{\partial \xi_m} - \frac{u_m}{R_m}\right), \qquad (m = 1, 2). \tag{33}$$

In this way the displacements are expressed—with little geometrical or analytical difficulty—in terms of the displacements of the middle surface.

To reduce the strain components to their final form we make use of (30) and (33) and of assumption 1° that the thickness of the shell is small compared with the radii of curvature.

We write

$$1 + \frac{\xi}{R_m} \approx 1 \tag{34}$$

and replace the values of A_m and their derivatives by their values on the middle surface, i. e.

$$A_m = \alpha_m, \quad \frac{\partial A_m}{\partial \zeta} = \frac{\alpha_m}{R_m}, \qquad (m = 1, 2). \tag{35}$$

The strain components (26) and (27) then become

$$\epsilon_1 = \frac{1}{\alpha_1}\frac{\partial}{\partial \xi_1}(u_1 + \zeta u'_1) + \frac{u_2 + \zeta u'_2}{\alpha_1 \alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w}{R_1} \tag{36}$$

$$\epsilon_2 = \frac{1}{\alpha_2}\frac{\partial}{\partial \xi_2}(u_2 + \zeta u'_2) + \frac{u_1 + \zeta u'_1}{\alpha_1 \alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} + \frac{w}{R_2} \tag{37}$$

$$\gamma_{12} = \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2 + \zeta u'_2}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1 + \zeta u'_1}{\alpha_1}\right) \tag{38}$$

which may be written

$$\epsilon_1 = \epsilon_1{}^0 + \zeta\kappa_1, \qquad \epsilon_2 = \epsilon_2{}^0 + \zeta\kappa_2, \qquad \gamma_{12} = \gamma_{12}{}^0 + \zeta\tau. \tag{39}$$

In (39) the first terms clearly give the strains of the middle surface and the second terms the amount of bending of the middle surface.

Introducing (36) to (38) into (20) to (23) and observing (34) there follows, with u' from (32),

$$\frac{1-\nu^2}{Eh}N_{11} = \epsilon_1{}^0 + \nu\epsilon_2{}^0 = \frac{1}{\alpha_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w}{R_1} + \nu\left(\frac{1}{\alpha_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} + \frac{w}{R_2}\right) \tag{40}$$

$$\frac{1}{Gh}N_{21} = \frac{1}{Gh}N_{12} = \gamma_{12}{}^0 = \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha_1}\right) \tag{41}$$

$$\frac{1-\nu^2}{Eh}N_{22} = \epsilon_2{}^0 + \nu\epsilon_1{}^0 = \frac{1}{\alpha_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} + \frac{w}{R_2} + \nu\left(\frac{1}{\alpha_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w}{R_1}\right) \tag{42}$$

$$\frac{12(1-\nu^2)}{Eh^3}M_{11} = \kappa_1 + \nu\kappa_2 = \frac{1}{\alpha_1}\frac{\partial u'_1}{\partial \xi_1} + \frac{u'_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \nu\left(\frac{1}{\alpha_2}\frac{\partial u'_2}{\partial \xi_2} + \frac{u'_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}\right) \tag{43}$$

$$\frac{12}{Gh^3}M_{12} = \frac{12}{Gh^3}M_{21} = \tau = \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{u'_2}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(\frac{u'_1}{\alpha_1}\right) \tag{44}$$

$$\frac{12(1-\nu^2)}{Eh^3}M_{22} = \kappa_2 + \nu\kappa_1 = \frac{1}{\alpha_2}\frac{\partial u'_2}{\partial \xi_2} + \frac{u'_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} + \nu\left(\frac{1}{\alpha_1}\frac{\partial u'_1}{\partial \xi_1} + \frac{u'_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2}\right). \tag{45}$$

The derivation of these stress-strain relations seems simpler and more direct than those given before. This is due to the fact that one works with the general strain components for curvilinear orthogonal coördinates and introduces into them the assumption that the normal to the undeformed middle surface is deformed into the normal to the deformed middle surface. Then one introduces the assumption that the shell is "thin" which shortens the resultant formulae greatly. This last step which in some form is usually introduced into the shell equations need however not be taken.

When Eqs. (40) to (45) are introduced into the first five of Eqs. (18) and (19) there remain five equations for the five unknown u_1, u_2, w, Q_1, Q_2, the solution of which constitutes the analytical part of the theory.

6. Concluding Remarks. Solutions of the shell equations have so far been found for cylindrical shells and for shells of rotational symmetry under axi-symmetrical load. No mathematical difficulty arises in the case of circular cylindrical shells, at least for most of the interesting types of boundary conditions. The beginnings of a theory of the general cylindrical shell are to be found in a recent paper by A. A. Jakobsen [7]. The theory of shells of rotational symmetry is due to H. Reissner [8] and E. Meissner [9], the former giving the solution for spherical shells, the latter showing that H. Reissner's method was applicable to the entire class of shells of rotational symmetry. In addition to these results there is a solution for spherical shells with unsymmetrical distribution of load by E. Schwerin [10]. An account of the results of the theory of circular cylindrical shells and of shells of rotational symmetry may be found in Love's Treatise and in books by W. Fluegge [11], C. B. Biezeno and R. Grammel [12] and S. Timoshenko [13], who also give further references.

Massachusetts Institute of Technology.

BIBLIOGRAPHY.

1. A. E. H. Love, *Philosophical Transactions of the Royal Society of London*, vol. 179 (1888), p. 491.
2. E. Mathieu, *J. Ecole Polytechn.*, vol. 51 (1883).
3. H. Aron, *J. Reine Ang. Math.*, vol. 78 (1874), p. 138.
4. F. Krauss, *Mathematische Annalen*, vol. 101 (1929), p. 61.
5. E. Trefftz, *Z. Ang. Math. Mech.*, vol. 15 (1935), p. 101.
6. F. K. G. Odquist, *C. R. Ac. Franc.*, vol. 205 (1937), p. 205, p. 271.
7. A. A. Jakobsen, *D. Bauingenieur*, vol. 28 (1937), p. 418, p. 436.
8. H. Reissner, *Festschrift H. Mueller-Breslau*, Leipzig 1912, p. 181.
9. E. Meissner, *Physik. Z.*, vol. 14 (1913), p. 343.
10. E. Schwerin, Dissertation T. H., Berlin 1917, J. Springer, Berlin 1918.
11. W. Fluegge, *Statik u. Dynamik der Schalen*, J. Springer, Berlin 1934.
12. C. B. Biezeno and R. Grammel, *Technische Dynamik*, J. Springer, Berlin 1939.
13. S. Timoshenko, *Theory of Plates and Shells*, McGraw Hill, 1940

ORTHOGONAL POLYNOMIALS DEFINED BY DIFFERENCE EQUATIONS.*†

By Otis E. Lancaster.

1. Introduction: Many analogous properties of differential and difference equations have been studied. Here these analogies are extended to include some ideas relative to orthogonal solutions of difference equations.

Although some general theorems are given, the main study is confined to polynomial solutions of difference equations of the form

$$(1) \qquad (ax^2 + bx + c)\underset{h}{\Delta^2}y(x) + (dx + f)\underset{h}{\Delta}y(x) + \lambda y(x+h) = 0$$

where $h > 0$ is the interval of difference, a, b, c, d, and f are constants and λ is a parameter which is determined so as to insure polynomial solutions.[1] After making a new definition of an adjoint equation, called the L-adjoint difference equation, it is proved that every second order difference equation can be put in L-self-adjoint form. Then it is shown that the solutions corresponding to characteristic values of λ are orthogonal (in a sense to be defined) on some interval. Special properties of these orthogonal functions are developed. Included among them is a difference form and a recurrence relation for the polynomial solutions of (1). The results are shown to reduce, in the limit as $h \to 0$, to the known facts for differential equations. The general theory is illustrated by the polynomials analogous to the Legendre and the Hermite Polynomials which were studied by Jordan [2] and Greenleaf,[3] respectively, also by polynomials analogous to the Laguerre and Jacobi Polynomials.

2. Definition of S-orthogonal functions.[4] *A sequence of functions*

---

* Received April 13, 1939; Revised July 1, 1940.

† Presented to the American Mathematical Society, December, 1938.

[1] We employ the symbol $\underset{h}{\Delta}y$ to mean $\dfrac{y(x+h) - y(x)}{h}$.

[2] Charles Jordan, "Sur une série de polynomes dont chaque somme partielle représente la meilleure approximation d'un degré donné suivant la méthode des moindres carrés," *Proceedings of the London Mathematical Society*, vol. 20 (1921), pp. 297-325; C. Jordan, "Approximation and Graduation According to the Principle of Least Squares by Orthogonal Polynomials," *Annals of Mathematical Statistics*, vol. 3 (1932), pp. 257-357.

[3] H. E. H. Greenleaf, "Curve Approximation by Means of Functions Analogous to the Hermite Polynomials," *Annals of Mathematical Statistics*, vol. 3 (1932), pp. 204-256.

[4] This definition reduces to the one given by Gram if the interval $[\mu, \nu]$ is a multiple of the interval of difference. See J. P. Gram, "Ueber die Entwickelung reeller Functionen in Reihen mittelst der Methode der kleinster Quadrate," *Journal für Mathematik*, vol. 94 (1883), pp. 41-73.

$\phi_0(x), \phi_1(x), \phi_2(x), \cdots, \phi_n(x), \cdots$ is S-orthogonal on the interval $[\mu, \nu]$ with a weight function $g(x)$, if

$$\overset{\nu}{\underset{\mu}{S}}\, g(x)\phi_k(x)\phi_m(x)\underset{h}{\Delta}x = 0, \qquad k \neq m,$$

where the operation[5] *S is the inverse operation of $\underset{h}{\Delta}$; that is, if*

$$g(x)\phi_k(x)\phi_m(x)\underset{h}{\Delta}x = \underset{h}{\Delta}q(x),$$

then

$$\overset{\nu}{\underset{\mu}{S}}\, g(x)\phi_k(x)\phi_m(x)\underset{h}{\Delta}x = q(x)\,\Big|_{\mu}^{\nu} = q(\nu) - q(\mu).$$

3. *L*-adjoint difference equations. In the study of linear homogeneous differential equations, a necessary and sufficient condition for a function $v(x)$ to be an integrating factor of an n-th order equation $L(y) = 0$ is that $v(x)$ satisfy an n-th order differential equation $\bar{L}(v) = 0$. This differential equation $\bar{L}(v) = 0$, is called the adjoint equation. There exist three fundamental relations between a differential equation and its adjoint. First, the relation is a reciprocal relation, that is, if $\bar{L}(v) = 0$ is the adjoint of $L(y) = 0$ then $L(y) = 0$ is the adjoint of $\bar{L}(v) = 0$. Second, Lagrange's identity

$$v\,L(y) - y\,\bar{L}(v) = \frac{d}{dx}\{P(y, v)\}$$

is satisfied, where $P(x)$ is linear and homogeneous in $y, y', y'', \cdots, y^{(n-1)}$, as well as in $v, v', v'', \cdots, v^{(n-1)}$. Third, a differential equation may be self-adjoint, that is $L(y) = 0$ may be identical with its adjoint.

Unfortunately, a linear homogeneous difference equation and its classical adjoint difference equation do not possess the above mentioned properties. *The relation is not a reciprocal one, there is no exact analogue to Lagrange's identity and no equations are self-adjoint unless the coefficients are periodic functions of x.* Since these relations are so *fundamental* in the study of S-orthogonal functions, we are led to make a new definition for an adjoint equation.

Consider a linear homogeneous difference equation of even order,

$$(2) \qquad L(y) = p_0(x)y(x + 2nh) + p_1(x)y(x + \overline{2n-1}h) + \cdots + p_n(x)y(x + nh) + \cdots + p_{2n}(x)y(x) = 0.$$

[5] This definition of the definite sum differs by a constant from the value obtained by replacing x by b in the principal sum defined by Nörlund. Hence the sum exists for all functions which are summable in the Nörlund sense. N. E. Nörlund, "Differenzenrechnung," p. 43.

Suppose there exists a function $v(x)$ such that $v(x+nh)L(y)$ is a perfect difference. Then by virtue of the identities

$$
(3)\quad \begin{cases}
v(x+nh)p_{2n-i}(x)y(x+ih) = \\
\qquad h\underset{h}{\Delta}[v(x+\overline{n-1}h)p_{2n-i}(x-h)y(x+\overline{i-1}h)+\cdots \\
\qquad\qquad +v(x+\overline{2n-i}h)p_{2n-i}(x+\overline{n-i}h)y(x+nh)] \\
\qquad\qquad +y(x+nh)p_{2n-i}(x+\overline{n-i}h)v(x+\overline{2n-i}h), \qquad i>n \\
v(x+nh)p_{2n-i}(x)y(x+ih) = \\
\qquad y(x+nh)p_{2n-i}(x+\overline{n-i}h)v(x+\overline{2n-i}h), \qquad i=n \\
v(x+nh)p_{2n-i}(x)y(x+ih) = \\
\qquad -h\underset{h}{\Delta}[v(x+\overline{2n-i-1}h)p_{2n-i}(x+\overline{n-i-1}h)y(x+\overline{n-1}h) \\
\qquad\qquad +\cdots+v(x+nh)p_{2n-i}(x)y(x+ih)] \\
\qquad\qquad +y(x+nh)p_{2n-i}(x+\overline{n-i}h)v(x+\overline{2n-i}h), \qquad i<n:
\end{cases}
$$

we obtain

$$
(4)\qquad v(x+nh)L(y)-y(x+nh)\bar{L}(v)=\underset{h}{\Delta}\{P(v,y)\}.
$$

where

$$
(5)\quad \bar{L}(v)=p_{2n}(x+nh)v(x+2nh)+p_{2n-1}(x+\overline{n-1}h)v(x+\overline{2n-1}h) \\
+\cdots+p_n(x)v(x+nh)+\cdots+p_0(x-nh)v(x)
$$

and

$$
(6)\quad P(v,y)=h\sum_{i=n+1}^{2n} v(x+\overline{n-1}h)p_{2n-1}(x-h)y(x+\overline{i-1}h) \\
+\cdots+v(x+\overline{2n-i}h)p_{2n-i}(x+\overline{n-i}h)y(x+nh) \\
-h\sum_{i=0}^{i=n-1} v(x+\overline{2n-i-1}h)p_{2n-i}(x-\overline{n-i-1}h)\cdot \\
y(x+\overline{n-1}h)+\cdots+v(x+nh)p_{2n-i}(x)y(x+ih).
$$

DEFINITION: *The difference equation $\bar{L}(v)=0$ shall be called the L-adjoint difference equation of the even*[6] *order difference equation $L(y)=0$.*

THEOREM 1. *The L-adjoint relation is a reciprocal one, that is the L-adjoint of $\bar{L}(v)=0$ is $L(y)=0$.*

The proof of this theorem follows immediately from the identities (3).

The relation (4) is a direct analogue of Lagrange's identity for differential equations.

DEFINITION: *When a linear homogeneous difference equation is identical with its L-adjoint, it is L-self-adjoint.*

---

[6] The generalization of this definition so as to include odd order difference equations has been omitted, since it has no advantages over the classical definition of an adjoint equation.

THEOREM 2. *A necessary and sufficient condition that the difference equation of even order $L(y)=0$, be L-self-adjoint is*

$$p_i(x)=p_{2n-i}(x+\overline{n-i}h) \qquad (i=0,1,2,\cdots,n-1). \tag{7}$$

This is immediate from the definition of the L-adjoint equation.

THEOREM 3. *Every second order linear homogeneous difference equation can be put in L-self-adjoint form.*

Proof: First, a difference equation of the form

$$\underset{h}{\Delta}(w(x)\underset{h}{\Delta}y(x))+s(x)y(x+h)=0 \tag{8}$$

is L-self-adjoint. This follows from Theorem 2, for (8) is equivalent to

$$w(x+h)y(x+2h)-[w(x+h)+w(x)-h^2s(x)]\,y(x+h) \\ +w(x)y(x)=0.$$

Second, every linear homogeneous difference equation can be put in the form (8) by multiplying it by a certain factor. Given a difference equation[7]

$$q(x)\underset{h}{\Delta^2}y(x)+r(x)\underset{h}{\Delta}y(x)+u(x)y(x+h)=0, \tag{9}$$

if $t(x)$ is such a factor, then

$$\begin{aligned} t(x)q(x)\underset{h}{\Delta^2}y(x)+t(x)r(x)\underset{h}{\Delta}y(x) &= \underset{h}{\Delta}(\phi(x)\underset{h}{\Delta}y(x)) \\ &= \phi(x+h)\underset{h}{\Delta^2}y(x)+\underset{h}{\Delta}\phi(x)\underset{h}{\Delta}y(x). \end{aligned}$$

Hence,

$$\begin{cases} t(x)q(x)=\phi(x+h) \\ t(x)r(x)=\underset{h}{\Delta}\phi(x) \end{cases}$$

or,

$$t(x+h)q(x+h)-t(x)q(x)=ht(x+h)r(x+h).$$

$$\therefore\quad t(x)=\exp\left(S\frac{1}{h}\log\frac{q(x)}{q(x+h)-hr(x+h)}\underset{h}{\Delta}x\right), \\ q(x+h)-hr(x+h)\neq 0. \tag{10}$$

When $q(x+h)-hr(x+h)=0$ the equation (9) reduces to a first order difference equation. Q. E. D.

4. S-orthogonality of Solutions of a Second Order Difference Equation. Suppose there is an infinite sequence of distinct values of λ; λ_0, λ_1, $\lambda_2\cdots$,

---

[7] Since a difference equation may be written in two forms we take the liberty to use the form that is most convenient for the point in question.

$\lambda_n, \cdots$ such that for each of these values the linear L-self-adjoint difference equation

$$\underset{h}{\Delta}(w(x)\underset{h}{\Delta}y(x)) + s(x)\lambda y(x+h) = 0 \tag{11}$$

has a solution which satisfies given boundary conditions. If the solution corresponding to λ_n is denoted by $y_n(x)$, then

$$\underset{h}{\Delta}(w(x)\underset{h}{\Delta}y_n(x)) + s(x)\lambda_n y_n(x+h) \equiv 0$$
$$\underset{h}{\Delta}(w(x)\underset{h}{\Delta}y_m(x)) + s(x)\lambda_m y_m(x+h) \equiv 0.$$

Upon multiplying the first of these by $y_m(x+h)$ and the second by $y_n(x+h)$ and utilizing the formula

$$\underset{h}{\Delta}[u(x)v(x)] = v(x+h)\underset{h}{\Delta}u(x) + u(x)\underset{h}{\Delta}v(x),$$

we obtain

$$\begin{aligned} y_m(x+h)[w(x+h)\underset{h}{\Delta^2}y_n(x) + \underset{h}{\Delta}w(x)\underset{h}{\Delta}y_n(x)] \\ + s(x)\lambda_n y_n(x+h)y_m(x+h) = 0 \\ y_n(x+h)[w(x+h)\underset{h}{\Delta^2}y_m(x) + \underset{h}{\Delta}w(x)\underset{h}{\Delta}y_m(x)] \\ + s(x)\lambda_m y_m(x+h)y_n(x+h) = 0. \end{aligned}$$

Subtracting the second of these identities from the first and adding and subtracting the term $w(x+h)\underset{h}{\Delta}y_m(x+h)\underset{h}{\Delta}y_n(x+h)$ to this result, we have

$$\begin{aligned} \underset{h}{\Delta}[w(x)\{y_m(x+h)\underset{h}{\Delta}y_n(x) - y_n(x+h)\underset{h}{\Delta}y_m(x)\}] = \\ (\lambda_m - \lambda_n)s(x)y_n(x+h)y_m(x+h). \end{aligned}$$

Hence,

$$\begin{aligned} w(x)\{y_m(x+h)\Delta y_n(x) - y_n(x+h)\underset{h}{\Delta}y_m(x)\}\Big|_{\mu}^{\nu} \\ = (\lambda_m - \lambda_n)\overset{\nu}{\underset{\mu}{S}} s(x)y_n(x+h)y_m(x+h)\underset{h}{\Delta}x. \end{aligned} \tag{12}$$

Therefore, if the definite sum $\overset{\nu}{\underset{\mu}{S}} s(x)y_n(x+h)y_m(x+h)\underset{h}{\Delta}x$ exists, if $w(x)$ vanishes for $x = \mu$ and $x = \nu$, and if the functions $y_m(x+h)$, $\underset{h}{\Delta}y_n(x)$, $y_n(x+h)$ and $\underset{h}{\Delta}y_m(x)$ are finite for $x = \mu$ and $x = \nu$, then the solutions $y_0(x+h)$, $y_1(x+h), \cdots$ are S-orthogonal on the interval $[\mu, \nu]$ with respect to the weight function $s(x)$. Or, *the sequence of functions* $\{y_n(x)\}$ *is S-orthogonal on the interval* $[\mu + h, \nu + h]$ *with the weight function* $s(x-h)$.

5. Polynomial Solutions. Consider the linear homogeneous difference equation

$$(13)\qquad p_n(x)\underset{h}{\Delta^n}y(x) + p_{n-1}(x)\underset{h}{\Delta^{n-1}}y(x) + \cdots + p_0(x)y(x+j) = 0,$$

where $p_i(x)$ is a polynomial of degree $\leq i$, and j is any constant. Define

$$(14)\qquad \theta(\rho) = \rho!\left[\frac{a_{nn}}{(\rho-n)!} + \frac{a_{n-1\,n-1}}{(\rho-n+1)!} + \cdots + \frac{a_{00}}{\rho!}\right]$$

where

$$p_i(x) = a_{i0} + a_{i1}x + \cdots + a_{ii}x^i.$$

THEOREM 4. *A necessary and sufficient condition that* (13) *have a polynomial solution* ($\not\equiv 0$) *is that the equation*

$$(15)\qquad \theta(\rho) = 0$$

have a non-negative integral root. If there is a polynomial solution of degree m, then[8] $\theta(m) = 0$. *If k is the smallest non-negative integral root, there is a solution of degree k, and there is no solution of degree less than k.*

Proof. If one substitutes $y = x^m + b_{m-1}x^{m-1} + \cdots + b_0$ in equation (13), then a necessary and sufficient condition that this be a solution is that the coefficients of all powers of x on the left be zero. The coefficients of x^m, $x^{m-1}, \cdots$ are readily seen to have the form

$$\theta(m),\ b_{m-1}\theta(m-1) + c_{m-1},\ b_{m-2}\theta(m-2) + c_{m-2}, \cdots$$

respectively, where the c's are determined by the a_{ij}'s and m. Hence if there is a solution of degree m, then $\theta(m) = 0$. Conversely, suppose there is a non-negative integral root of $\theta(\rho) = 0$ and let k be the smallest such. Then $\theta(k-r) \neq 0$ for $r = 1, 2, \cdots, m$ but $\theta(k) = 0$. Hence the b's can be uniquely determined so that the above coefficients vanish. That is, there is at least one polynomial solution and one of the solutions is of degree k. Since $\theta(m)$ must be zero for a solution of degree m, it follows that there is no solution of degree less than k. Q. E. D.

The same argument leads to precisely the same conclusions regarding the corresponding differential equation

$$(16)\qquad p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_0(x)y(x) = 0.$$

From this fact follows at once the

COROLLARY 4.1. *If* (16) *has a polynomial solution so does* (13) *and conversely. Moreover the minimum degree of all polynomial solutions of* (13) *is the same as that of* (16). *In particular, if* (16) *has only one polynomial*

---

[8] If $\theta(m) = 0$, it does not, however, follow that there is a solution of degree m.

solution, then (13) *has a polynomial solution of the same degree and conversely.*[9]

Theorem 4 insures that (1) has a polynomial solution of the m-th degree, when m is the smallest integral value for which

$$am(m-1)+dm+\lambda=0. \tag{17}$$

So, when $d \neq ka$, k a positive integer,[10] there exists an infinite sequence of distinct characteristic values for $\lambda: \lambda_0, \lambda_1, \lambda_2 \cdots$ such that (1) is satisfied by a polynomial of m-th degree when

$$\lambda=\lambda_m=-am(m-1)-dm.$$

When (1) is written in its L-self-adjoint form (11)

---

[9] In a way, this result is very surprising, for in general the solutions of difference and differential equations are quite different in nature. It is important to note precisely the statement of the corollary. Although there is always a polynomial solution of (13) that is of the same degree as a polynomial solution of (16), the two solutions are not the same. Moreover (16) may have more polynomial solutions than (13) and conversely. For example

$$(x^2-\tfrac{4}{3}x+3)y''+(-6x+4)y'+12y=0$$

is satisfied by

$$y=x^4+18x^2+16x+\tfrac{11}{3},$$

but

$$(x^2-\tfrac{4}{3}x+3)\underset{h}{\Delta^2}y(x)+(-6x+4)\underset{h}{\Delta}y(x)+12y=0$$

is not satisfied by any polynomial of the fourth degree.

The corollary cannot hold if the degree of $p_i(x)$ is greater than i. For example:

$$(x^3+3x^2)y''-(12x-36)y=0$$

is satisfied by $y=x^4$ but

$$(x^3+3x^2)\Delta^2y(x)-(12x-36)y=0$$

does not have a polynomial solution.

Moreover it is evident that the corollary could not be extended to include non-linear equations. For, if a non-linear algebraic difference equation of q-th degree and n-th order has a polynomial solution of degree m, then the m coefficients of the polynomial must satisfy $mn-1$ relations and in general this is not possible. For example:

$$(y')^2-4y=0$$

has a polynomial solution $y=x^2$ but

$$[\Delta y(x)]^2-4y=0$$

does not have a solution of the form

$$y=a_0+a_1x+a_2x^2.$$

[10] If $d=-ka$ then λ_n may equal λ_m, $n\neq m$. And if $n>m$ there may not be a polynomial solution of degree n. For a relation equivalent to (17), see E. H. Hildebrandt, "Systems of Polynomials Connected with the Charlier Expansions and the Pearson Differential and Difference Equations," *Annals of Mathematical Statistics*, vol. 2 (1931), p. 405.

$$(18)\ w(x) = [a(x-h)^2 + b(x-h) + c] \exp(S \frac{1}{h} \frac{a(x-h)^2 + b(x-h) + c}{ax^2 + bx + c - h(dx+f)} \underset{h}{\Delta}$$

Thus, if x_1 and x_2 are two real zeros of $w(x)$, the results of section 4 show that the sequence of polynomials $\{y_m(x)\}$ are S-orthogonal on the interval $[x_1 + h, x_2 + h]$ with the weight function

$$(19) \qquad g(x) = \exp(S \frac{1}{h} \log \frac{a(x-h)^2 + b(x-h) + c}{ax^2 + bx + c - h(dx+f)} \underset{h}{\Delta} x).$$

provided that the definite sum of this weight function on that interval has a meaning. We see, moreover, that if the weight function (21) is different from zero everywhere, the interval of S-orthogonality is $[\mu, \nu]$, where $\mu - 2h$ and $\nu - 2h$ are the roots of the equation

$$(20) \qquad ax^2 + bx + c = 0.$$

6. Difference Form of the Polynomial Solutions of (1). If we set

$$y(x) = \underset{h}{\Delta}^{n+1} z(x)$$

the equation (1) becomes

$$(21) \quad (ax^2 + bx + c)\underset{h}{\Delta}^{n+3} z(x) + (dx+f)\underset{h}{\Delta}^{n+2} z(x) + \lambda \underset{h}{\Delta}^{n+1} z(x+h) = 0.$$

And upon summing $n+1$ times by means of the generalized Leibnitz theorem for summation,

$$(22) \quad \underset{h}{\Delta}^{-n} u(x)v(x) = u(x-h)\underset{h}{\Delta}^{-n} v(x) - \frac{n}{1!} \underset{h}{\Delta} u(x - \overline{n-1}h)\underset{h}{\Delta}^{-n-1} v(x)$$
$$+ \frac{n(n+1)}{2!} \underset{h}{\Delta}^2 u(x - \overline{n-2}h)\underset{h}{\Delta}^{-n-2} v(x) \cdots;$$

we obtain [11]

$$\begin{aligned} &[a(x - \overline{n+1}h)^2 + b(x - \overline{n+1}h) + c]\underset{h}{\Delta}^2 z(x) \\ (23) \qquad &+ [dx + f + (n+1)^2 ah - (n+1)(2ax+b)]\underset{h}{\Delta} z(x) \\ &+ [(n+1)(n+2)a - (n+1)d + \lambda] z(x+h) = 0. \end{aligned}$$

If

$$(24) \qquad \lambda - (n+1)d + (n+1)(n+2)a = 0$$

this reduces to

---

[11] Since the operators $\underset{h}{\Delta}^{-k}$ (k = positive integer) of (22) lack uniqueness, the right hand side of (23) should be some function $f(x)$ whose n-th difference is 0. If however we choose $f(x) = 0$ and use (23) to define $Z(x)$, then by differencing we obtain (21), so that $y(x) = \underset{h}{\Delta}^{n+1} z(x)$ is a solution of (1). Hence we may take (23) in its homogeneous form.

$$[a(x-\overline{n+1}h)^2+b(x-\overline{n+1}h)+c]\underset{h}{\Delta^2}z(x)$$
$$+[dx+f+(n+1)^2ah-(n+1)(2ax+b)]\underset{h}{\Delta}z(x)=0.$$

And by a simple summation, we have

$$\underset{h}{\Delta}z(x)=\exp(-S\frac{1}{h}\log\frac{a(x-\overline{n+1}h)^2+b(x-\overline{n+1}h)+c}{ax^2+bx+c-h(dx+f)}\underset{h}{\Delta}x).$$

Hence,

$$(25)\qquad y(x)=\underset{h}{\Delta^n}[\exp(S\frac{1}{h}\log\frac{a(x-\overline{n+1}h)^2+b(x-\overline{n+1}h)+c}{ax^2+bx+c-h(dx+f)}\underset{h}{\Delta}x)].$$

Now, if $v(x)$ is a solution of the L-adjoint equation of (1), then

$$v(x+h)[(ax^2+bx+c)\underset{h}{\Delta^2}y(x)+(dx+f)\underset{h}{\Delta}y(x)+\lambda y(x+h)]$$
$$=\underset{h}{\Delta}[M(x)\underset{h}{\Delta}y(x)+H(x)y(x)],$$

where

$$M(x)=v(x)[a(x-h)^2+b(x-h)+c]$$
$$hH(x)=hv(x+h)(dx+f)+v(x)[a(x-h)^2+b(x-h)+c]$$
$$-v(x+h)(ax^2+bx+c).$$

Therefore, a first summation of the equation (1) is

$$M(x)\underset{h}{\Delta}y(x)+H(x)y(x)=0.$$

Whence,

$$y(x)=\exp(S\frac{1}{h}\log\frac{v(x+h)(ax^2+bx+c)-(dx+f)hv(x+h)}{v(x)[a(x-h)^2+b(x-h)+c]}\underset{h}{\Delta}x)$$

or

$$(26)\qquad y(x)=v(x)\exp(S\frac{1}{h}\log\frac{ax^2+bx+c-(dx+f)h}{a(x-h)^2+b(x-h)c}\underset{h}{\Delta}x).$$

The adjoint of (1), as determined by the method of section 3. is

$$(27)\qquad [a(x+h)^2+(b-hd)(x+h)+c-fh]\underset{h}{\Delta^2}v(x)$$
$$+[(4a-d)x+(2b-f-dh)]\underset{h}{\Delta}v(x)+[2a-d+\lambda]v(x+h)=0.$$

This difference equation is of the form (1); hence it has a solution of the form (25) when a condition analogous to (24) is satisfied: that is, if

$$(2a-d+\lambda)-(n+1)(4a-d)+(n+1)(n+2)a=0$$

or, what is equivalent,

$$(28)\qquad n(n-1)a+dn+\lambda=0.$$

Thus, when (28) is satisfied by an integer n,

$$v_n(x) = \underset{h}{\Delta^n}[\exp(-S\frac{1}{h}\log\frac{a(x-nh)^2+(b-dh)(x-nh)+c-hf}{a(x-h)^2+b(x-h)+c}\underset{h}{\Delta x})]$$

and

$$(29)\quad y_n(x) = \exp(S\frac{1}{h}\log\frac{ax^2+bx+c-h(dx+f)}{a(x-h)^2+b(x-h)+c}\underset{h}{\Delta x})$$
$$\times \underset{h}{\Delta^n}[\exp(S\frac{1}{h}\log\frac{a(x-h)^2+b(x-h)+c}{a(x-nh)^2+(b-dh)(x-nh)+c-hf}\underset{h}{\Delta x})].$$

The condition that $\lambda = -n(n-1)a - nd$, where n is an integer was sufficient to insure that (1) has a polynomial solution. Can (29) be a representation of these polynomial solutions?[12] We shall see that the answer is in the affirmative.

In order to discuss the various cases which arise for the solutions (29), it is convenient to let $\Gamma_h(x)$ denote a solution of the difference equation

$$s(x+h) - xs(x) = 0;$$

α_1 and α_2 denote the roots of the equation

$$a(x-h)^2 + b(x-h) + c = 0;$$

and β_1 and β_2 denote the roots of the equation

$$ax^2 + (b-dh)x + c - hf = 0.$$

Case I. $a \neq 0$. In terms of the above notation, (29) may be written as

$$y_n(x) = \frac{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)}{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}\underset{h}{\Delta^n}\frac{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}{\Gamma_h(x-\beta_1-nh)\,\Gamma_h(x-\beta_2-nh)}$$

or

$$(30)\quad y_n(x) = \frac{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)}{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}$$
$$\times \underset{h}{\Delta^n}\left[(x-h-\beta_1)^{(nh)}(x-h-\beta_2)^{(nh)}\frac{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)}\right].$$

This expression is a polynomial for any positive integer n. Hence, it is the difference expression for the polynomial solutions of (1).

Case II. $a = 0,\ b \neq 0,\ b - dh \neq 0$.

$$(31)\quad y_n(x) = \frac{\Gamma_h(x-\beta_1)}{\Gamma_h(x-\alpha_1)}\left(\frac{b-dh}{b}\right)^{x/h}$$
$$\times \underset{h}{\Delta^n}\left[(x-h-\beta_1)^{(nh)}\frac{\Gamma_h(x-\alpha_1)}{\Gamma_h(x-\beta_1)}\left(\frac{b}{b-dh}\right)^{x/h}\right].$$

---

[12] When $h = 1$, the solution (29) can be identified as the polynomials $Q_n(n, x)$ of Hildebrandt's paper referred to in footnote 10. The identification is made evident when one observes that $t(x)$ is a solution of a difference equation of type (1) on page 421 and that $Q_n(n, x)$ is a solution of the second order difference equation XIV_n on page 433.

Case III. $a=0,\ b=0,\ d\neq 0,\ c\neq 0.$

$$(32)\qquad y_n(x)=\Gamma_h(x-\beta_1)\left(-\frac{dh}{c}\right)^{x/h}$$
$$\times\ \underset{h}{\Delta^n}\left[(x-h-\beta_1)^{(nh)}\frac{1}{\Gamma_h(x-\beta_1)}\left(-\frac{c}{dh}\right)^{x/h}\right].$$

Case IV. $a=0,\ b\neq 0,\ b-dh=0,\ c-hf\neq 0.$

$$(33)\qquad y_n(x)=\frac{1}{\Gamma_h(x-\alpha_1)}\left(\frac{c-hf}{b}\right)^{x/h}\underset{h}{\Delta^n}\left[\Gamma_h(x-\alpha_1)\left(\frac{b}{c-hf}\right)^{x/h}\right].$$

In the last three cases the expressions for $y_n(x)$ are also polynomials for integral values of n. Hence, we do have the difference form of the polynomials of (1).[13]

7. Recurrence Relation:

THEOREM 5. *The polynomials $y_n(x)$ satisfy a recurrence relation of the form*

$$(34)\qquad A(n)y_{n+2}(x)+[B(n)x+C(n)]y_{n+1}(x)+D(n)y_n(x)=0.$$

Proof: The proof is divided into four cases corresponding to the four forms of $y_n(x)$; (30), (31), (32), and (33).

Case I. If the theorem is true for $y_n(x)$ defined by (30), then

$$A(n)\underset{h}{\Delta^{n+2}}\left[(x-h-\beta_1)^{(\overline{n+2}h)}(x-h-\beta_2)^{(\overline{n+2}h)}\frac{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)}\right]$$
$$+[B(n)x+C(n)]$$
$$\times\underset{h}{\Delta^{n+1}}\left[(x-h-\beta_1)^{(\overline{n+1}h)}(x-h-\beta_2)^{(\overline{n+1}h)}\frac{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)}\right]$$
$$+D(n)\underset{h}{\Delta^{n}}\left[(x-h-\beta_1)^{(nh)}(x-h-\beta_2)^{(nh)}\frac{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)}\right]=0,$$

which, after n summations, yields (the particular result)

$$A(n)\underset{h}{\Delta^{2}}\left[(x-h-\beta_1)^{(\overline{n+2}h)}(x-h-\beta_2)^{(\overline{n+2}h)}\frac{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)}\right]$$
$$+[B(n)x+C(n)]$$
$$\times\underset{h}{\Delta}\left[(x-h-\beta_1)^{(\overline{n+1}h)}(x-h-\beta_2)^{(\overline{n+1}h)}\frac{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)}\right]$$
$$-nB(n)(x-\beta_1)^{(\overline{n+1}h)}(x-\beta_2)^{(\overline{n+1}h)}\frac{\Gamma_h(x-\alpha_1+h)\,\Gamma_h(x-\alpha_2+h)}{\Gamma_h(x-\beta_1+h)\,\Gamma_h(x-\beta_2+h)}$$
$$+D(n)(x-\beta_1-h)^{(nh)}(x-\beta_2-h)^{(nh)}\frac{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)}=0.$$

---

[13] This constitutes a second proof that (1) has a polynomial solution when

$$\lambda=-n(n-1)a-nd.$$

The first theorem is given because it is of interest in itself.

After effecting these differences and removing the common factor

$$(x-h-\beta_1)^{(nh)}(x-h-\beta_2)^{(nh)}\frac{\Gamma_h(x-\alpha_1)\,\Gamma_h(x-\alpha_2)}{\Gamma_h(x-\beta_1)\,\Gamma_h(x-\beta_2)},$$

we obtain

$$\begin{aligned} A(n)\{&(x+h-\alpha_1)(x-\alpha_1)(x+h-\alpha_2)(x-\alpha_2)-2(x-\alpha_1)(x-\alpha_2)\\ &\qquad(x-\overline{n+1}h-\beta_1)(x-\overline{n+1}h-\beta_2)\\ &+(x-\overline{n+1}h-\beta_1)(x-\overline{n+1}h)(x-\overline{n+2}h-\beta_2)(x-\overline{n+2}h-\beta_2)\\ +B(n)\{&hx(x-\alpha_1)(x-\alpha_2)-hx(x-\overline{n+1}h-\beta_1)(x-\overline{n+1}h-\beta_2)\\ &\qquad-h^2n(x-\alpha_1)(x-\alpha_2)\}\\ +C(n)\{&h(x-\alpha_1)(x-\alpha_2)-h(x-\overline{n+1}h-\beta_1)(x-\overline{n+1}h-\beta_2)\}\\ &\qquad+h^2D(n)=0. \end{aligned}$$

If this relation is to be satisfied for all values of x, the coefficients of the powers of x must vanish. The coefficients of x^4 and x^3 vanish for all values of A, B, C, D. Equating the cofficients of x^2, x^1 and x^0 to zero we obtain three linear homogeneous equations in the four unknowns. Hence there is always a non-trivial solution.

When A is chosen so as to avoid fractions as much as possible a solution is [14]

$$(35)\quad\left\{\begin{aligned} A&=-(n+d/a)(2n+d/a)\\ B&=(2n+d/a)(2n+1+d/a)(2n+2+d/a)\\ C&=-(d/a-2)(2n+1+d/a)\left(\frac{bd-2af}{2a^2}\right)+(b/2a)B\\ &\quad-(d/a-2)(2n+1+d/a)(n+1+nd/2a+d/a)h\\ &\qquad-(2n+1+d/a)(2n+2+d/a)(2n+1+nd/2a+d/a)h\\ D&=(n+1)(2n+2+d/a)\left\{\left(\frac{bd-2af}{2a^2}\right)^2-(2n+d/a)^2\frac{b^2-4ac}{4a^2}\right\}\\ &\quad+n(n+1)(n+d/a)(2n+2+d/a)\frac{bd-2af}{a^2}h\\ &\qquad+n^2(n+1)(n+d/a)(2n+2+d/a)h^2. \end{aligned}\right.$$

Cases II, III, and IV. By carrying through exactly the same steps as in Case I we can show that there is always a non-trivial solution for A, B, C, and D.

For Case II, where $y_n(x)$ is defined by (31), a solution for A, B. C, and D is

---

[14] The author wishes to thank Mr. W. S. Cramer for checking the evaluations for A, B, C, and D.

To obtain these values for A, B, C, and D it is convenient to obtain the three relations by letting x equal α_1, $(n+1)h+\beta_1$ and $(n+1)h+\beta_2$.

Although (35) was obtained by assuming that (34) is true, it is clear that the steps can be retraced, so that from (35) follows the truth of (34). The same remark applies to the other cases.

$$(36) \qquad \begin{cases} A = dh - b \\ B = d \\ C = (2n+2)d + f - d(n+2)h \\ D = (n+1)\left[\dfrac{cd - fb}{b - dh} - nb\right]. \end{cases}$$

A solution for Case III, where instead of (32), we have

$$y_n(x) = h^n \Gamma_h(x - \beta_1)\left(\frac{-dh}{c}\right)^{x/h} \times \underset{h}{\Delta^n}\left[(x - h - \beta_1)^{(nh)} \frac{1}{\Gamma_h(x - \beta_1)}\left(\frac{-c}{dh}\right)^{x/h}\right],$$

is

$$(37) \qquad \begin{cases} A = 1, \quad B = 1, \quad C = -(n+2)h + f/d \\ D = -(n+1)c/d \end{cases}$$

And a solution for Case IV, where $y_n(x)$ is defined by (33) is

$$(38) \qquad \begin{cases} A = fh - c, \qquad B = d \\ C = f + bn, \qquad D = (n+1)d \end{cases}$$

8. Orthogonality and Zeros of the Polynomials.[15] In this discussion, we first consider the last three cases.

Case II. The weight function (19), which in this case is

$$\frac{\Gamma_h(x - \alpha_1)}{\Gamma_h(x - \beta_1)}\left(\frac{b}{b - dh}\right)^{x/h},$$

has poles at the points $\alpha_1, \alpha_1 - h, \alpha_1 - 2h, \cdots$ and zeros at $\beta_1, \beta_1 - h, \beta_1 - 2h, \cdots$. Thus, if $bd < 0$, the sequence of polynomials is S-orthogonal on the interval $[\alpha_1 + h, \infty]$. For then, $b(b - dh) > 0$, $w(x)$ vanishes at the two points $x = \alpha + h$ and $x = \infty$ and the weight function is summable over this interval. And, if $bd > 0$, $b > dh$ and if $\alpha_1 = \beta_1 - kh$, $(k > 0)$, then the polynomials are S-orthogonal on the interval $(-\infty, \alpha + h]$.

THEOREM 6. *If $bd < 0$, the zeros of the polynomials of Case II are real distinct and lie on the interval $[\alpha + h, \infty]$. If $bd > 0$, $b > dh$ and $\alpha_1 = \beta_1$, the zeros of the polynomials are real distinct and lie on the interval $(-\infty, \alpha + h]$. Moreover, if $fd > cd$ and either $bd < 0$, or $bd > 0$ and $b > dh$ then the zeros of $y_n(x)$ separate those of $y_{n-1}(x)$.*

LEMMA. *If $P_0, P_1, P_2, \cdots, P_n, \cdots$ form a sequence of S-orthogonal*

[15] Note that the difference form and the recurrence relation for the polynomials hold whether the polynomials are S-orthogonal or not, that is, even if the weight function is not summable over the interval in question.

polynomials, and if $P_n(x)$ is of the n-th degree, then P_n is S-orthogonal to any polynomial of degree less than n.

The proof of this lemma follows immediately from the fact that any polynomial may be expressed as a sum of the P's.

Proof of the theorem. If $fb > cd$, and either $bd < 0$, or $bd > 0$ and $b > dh$, then in the recurrence relation (34), $AD > 0$ for all n, hence the polynomials form a generalized Sturmian sequence.[16] Therefore, the zeros of $y_n(x)$ are real, distinct, lie on the interval of orthogonality and separate those of $y_{n-1}(x)$.

If $fb \leq cd$ and $bd < 0$, then $\beta_1 < \alpha_1 + h$ and the weight function is positive over the entire interval $[\alpha + h, \infty)$. Hence, if

$$\mathop{S}_{a+h}^{\infty} y_n(x) g(x) \mathop{\Delta x}_{h} = 0,$$

there is at least one zero of $y_n(x)$ on the interval. Assume that there are less than n; then

$$y_n(x) = (x - w_1)(x - w_2) \cdots (x - w_k) W(x), \qquad (k < n),$$

where $W(x)$ a polynomial of degree $n - k$, which has no real zeros. By the lemma

$$\mathop{S}_{a+h}^{\infty} y_n(x)(x - w_1)(x - w_2) \cdots (x - w_k) g(x) \mathop{\Delta x}_{h} = 0 \qquad (g(x) > 0),$$

or

$$\mathop{S}_{a+h}^{\infty} (x - w_1)^2 (x - w_2)^2 \cdots (x - w_k)^2 W(x) g(x) \mathop{\Delta x}_{h} = 0.$$

The last statement cannot hold unless $W(x)$ vanishes on the interval. This contradicts the assumption that $W(x)$ has no real roots. Hence all roots of $y_n(x)$ are real and lie on the interval $[\alpha + h, \infty)$.

Suppose that they are not distinct, then

$$y_n(x) = (x - w_1)^{m_1} (x - w_2)^{m_2} \cdots (x - w_k)^{m_k}.$$

Let $Z(x)$ represent a product formed by taking one factor from each of the factors of odd multiplicity. Now $Z(x)$ is of degree less than that of $y_n(x)$, so

$$\mathop{S}_{a+h}^{\infty} y_n(x) Z(x) g(x) \mathop{\Delta x}_{h} = 0.$$

[16] See M. B. Porter, "On the Roots of Functions Connected by a Linear Recurrent Relation of the Second Order," *Annals of Mathematics* (2nd series), vol. 3 (1901-1902), pp. 55-70.

Again the function in the summation is of one sign, hence its sum cannot vanish. Thus the roots are all distinct.

If $bd > 0$, $b > dh$, $fd \leq cd$ and if $\alpha_1 = \beta_1$ then a similar argument holds concerning the zeros of the solutions. Q. E. D.

Case III. If $cd < 0$ the weight function is real and summable over any interval. Hence, the polynomials are S-orthogonal over the interval

$$[\beta - jh, \infty) \qquad (j = 0, 1, 2, \cdots)$$

Moreover, when $cd < 0$, $AD > 0$, hence the zeros of $y_n(x)$ are real, distinct, lie on the interval $[\beta, \infty)$ and separate those of $y_{n-1}(x)$.

Case IV. If $d(fh - c) > 0$, then in the recurrent relation (34), $AD > 0$, so the polynomials form a generalized Sturmian sequence. Therefore the zeros of $y_n(x)$ are real, distinct, and separate those of $y_{n-1}(x)$. Since $w(x)$ vanishes at only one point the polynomials are not S-orthogonal.

Case I. The weight function

$$\frac{\Gamma_h(x - \alpha_1)\,\Gamma_h(x - \alpha_2)}{\Gamma_h(x - \beta_1)\,\Gamma_h(x - \beta_2)}.$$

has zeros at

$$x = \beta m + jh \qquad (m = 1, 2; j = 0, 1, 2, \cdots).$$

and poles at the points

$$x = \alpha_m + jh \qquad (m = 1, 2; j = 0, 1, 2, \cdots).$$

If our weight function is to be finite over the interval of summation then either

1) $\alpha_1 + h > \alpha_2$,

or

2) $\beta_m > \alpha_2 + h \qquad (m = 1, 2)$.

or

3) $\alpha_2 = \beta_m - jh \qquad (m = 1, 2;\ j = 0, 1, 2, \cdots)$.

In cases 1) and 2) the polynomials are S-orthogonal over an interval of length less than h, so are of little importance from a practical standpoint.

If

$$\alpha_2 = \beta_m - jh \qquad (m = 1, 2; j = 0, 1, 2, \cdots)$$

the interval of S-orthogonality may be of any length (α_1, may be any number. $\alpha_1 < \alpha_2$) and the weight function is

$$(x - \beta_m + \overline{j - 1}h)^{(jh)} \frac{\Gamma_h(x - \alpha_1)}{\Gamma_h(x - \beta_c)}, \qquad (e \neq m = 1, 2).$$

If, moreover, $\alpha_1 = \beta_e - ih$, then the weight function is

$$(39) \qquad (x - \beta_m + \overline{j - 1}h)^{(jh)} (x - \beta_e + \overline{i - 1}h)^{(ih)}.$$

It follows immediately from (35) that these polynomials do not satisfy Porter's conditions which are sufficient to insure a generalized Sturmian sequence, for regardless of the values of the constants a, b, c, d and f and the magnitude of h, if n is sufficiently large $AD < 0$. This does not prove a thing, but it suggests that the zeros of the polynomials may not be real, distinct, etc. Upon examining the first five polynomials of a special case of the polynomials treated by Jordan, viz., the solutions of

$$(x^2 + 4x + 3)\Delta^2 y(x) + (2x + 3)\Delta y(x) - n(n + 1)y(x + 1) = 0,$$

namely,

$$U_0 = 1, \quad U_1 = x + 1/2, \quad U_2 = 3/2(x^2 + x),$$
$$U_3 = 5/2(x^3 + 3/2x^2 + 1/2x), \quad U_4 = x^4 + 2x^3 + 11/7x^2 + 4/7x;$$

we see that this is the case, for U_4 has two complex roots. Many other examples exhibit the same properties. So, we may state that *in general the roots of the polynomials of Case I are not all real.*[17]

9. The evaluation of $\overset{\nu}{\underset{\mu}{S}}\, y_n^2(x) g(x) \underset{h}{\Delta} x$. Let the polynomials be S-orthogonal on the interval $[\mu, \nu]$ with a weight function $g(x)$. Multiplying equation (34) by $y_n(x)g(x)$ and summing from μ to ν we obtain:

$$B(n) \overset{\nu}{\underset{\mu}{S}}\, x y_{n+1}(x) y_n(x) g(x) \underset{h}{\Delta} x + D(n) \overset{\nu}{\underset{\mu}{S}}\, y_n^2(x) g(x) \underset{h}{\Delta} x = 0.$$

Upon reducing the subscripts of (34) by one, multiplying the relation by $y_{n+1}(x)g(x)$ and summing from μ to ν, we have

$$A(n-1) \overset{\nu}{\underset{\mu}{S}}\, y^2_{n+1}(x) g(x) \underset{h}{\Delta} x + B(n-1) \overset{\nu}{\underset{\mu}{S}}\, x y_n(x) y_{n+1}(x) g(x) \underset{h}{\Delta} x = 0.$$

Whence, if

---

[17] It should be noted that this is not contrary to the statement made by L. Fejér in a note at the close of the first of the two papers by Jordan mentioned in footnote 1. Fejér proved that $y_m(x)$, a solution of

$$(x - a + 2h)(x - b + 2h)\underset{h}{\Delta^2} y_m(x) + [2x - a - b + 3h - m(m + 1)h]\underset{h}{\Delta} y_m(x) - m(m + 1) y_m(x) = 0$$

has real, distinct zeros which lie on the interval $[a, b - h]$ *provided* $m < \dfrac{b - a}{h}$. This provision was not strongly emphasized in the paper of Jordan in the *Annals of Mathematical Statistics*, vol. 3 (1932).

$$B(n)A(n-1) \neq 0$$

$$\overset{\nu}{\underset{\mu}{S}}\, y^2_{n+1}(x)g(x)\underset{h}{\Delta x} = \frac{B(n-1)D(n)}{B(n)A(n-1)}\overset{\nu}{\underset{\mu}{S}}\, y_n^2(x)g(x)\underset{h}{\Delta x}$$

Therefore, if $A(n)$, $B(n)$ and $D(n)$ are different from zero for all we may evaluate

$$\overset{\nu}{\underset{\mu}{S}}\, y_n^2(x)g(x)\underset{h}{\Delta x}.$$

For

$$\overset{\nu}{\underset{\mu}{S}}\, y_n^2(x)g(x)\underset{h}{\Delta x} \tag{40}$$
$$= \frac{B(0)}{B(n-1)}\cdot\frac{D(n-1)D(n-2)\cdots D(0)}{A(n-2)A(n-3)\cdots A(-1)}\overset{\nu}{\underset{\mu}{S}}\, y_0^2(x)g(x)\underset{h}{\Delta x}$$

and

$$\overset{\nu}{\underset{\mu}{S}}\, y_0^2(x)g(x)\underset{h}{\Delta x}$$

may be found by direct summation.

10. Examples. To illustrate the above general theory we consider some special cases.

a. *Analogue to Legendre polynomials:* The polynomials studied by Jordan satisfy the difference equation.

$$(x-a+2h)(x-b+2h)\Delta^2U_m(x)$$
$$+ [2x-a-b+3h-m(m+1)h]h\Delta U_m(x) - m(m+1)h^2U_m(x) = 0.$$

which may be written in the form

$$(x-a+2h)(x-b+2h)\underset{h}{\Delta^2}U_m(x)$$
$$+ [2x-a-b+3h]\underset{h}{\Delta}U_m(x) - m(m+1)U_m(x+h) = 0.$$

Formulas (19) and (20) show that the polynomials are S-orthogonal on the interval $[a,b]$ with a weight function 1; formula (30) gives

$$U_m(x) = \underset{h}{\Delta^m}[(x-a)^{(mh)}(x-b)^{(mh)}].$$

and from the values (35) we obtain

$$-U_{m-2}(x) + (2m+3)(2x-a-b+h)U_{m+1}(x)$$
$$-2(m+1)^2[a-b-(m+1)^2h^2]U_m(x) = 0.$$

When we take into consideration that the polynomials $Q_m(x)$ of Jordan are multiplied by the factor $\dfrac{1}{2^m m!}$, we obtain his recurrent relation:

$$4(m+2)Q_{m+2}(x)-2(2m+3)(2x-a-b+h)Q_{m+1}$$
$$+(m+1)[(b-a)^2-(m+1)^2h^2]Q_m=0.$$

If $b-a=nh$, formula (40) yields

$$\underset{a}{\overset{b}{S}}\, Q_m^2(x)\underset{h}{\Delta x}=\frac{nh^{2m+1}}{4^m(2m+1)}(n^2-1)(n^2-2^2)(n^2-3^2)\cdots(n^2-m^2).$$

b. *Greenleaf's analogue to the Hermite Polynomials:* Greenleaf studied the polynomials satisfying the difference equation

$$(p-x-1)\Delta^2\phi_n(x)+2(n-x-1)\Delta\phi_n(x)+2n\phi_n(x)=0$$

or

$$(p-x-1)\Delta^2\phi_n(x)-2(x+1)\Delta\phi_n(x)+2n\phi(x+1)=0.$$

From (18) we obtain

$$w(x)=(p-x)\exp(S\log\frac{p-x}{p+x+1}\Delta x)$$
$$=(p-x)\exp(S\log\frac{(p-x-1)!\,(p+x)!}{(p-x-1)!\,(p+x)!}\frac{(p-x)}{(p+x+1)}\Delta x)$$
$$=(p-x)\frac{1}{(p-x)!\,(p+x)!}$$

and since $0!=1$, $w(x)=0$ when $x=p$ and $x=-p-1$.
Hence,

$$\underset{-p}{\overset{p+1}{S}}\,\phi_n\phi_m\frac{1}{(p-x)!\,(p+x)!}\Delta x=0,$$

or

$$\underset{-p}{\overset{p+1}{S}}\,\phi_n\phi_m\frac{2p!}{2^{2p}(p-x)!\,(p+x)!}\Delta x=0.$$

The formula (29) yields

$$\phi_n(x)=(p-x)!\,(p+x)!\,\Delta^n\Big[(p+x-1)^{(n)}\frac{1}{(p-x)!\,(p+x)!}\Big].$$

Taking into consideration the factor $(-\frac{1}{2})^n$ in Greenleaf's polynomials, we obtain from the values (36) the recurrent relation

$$4\phi_{n+2}(x)-4x\phi_{n+1}(x)+(n+1)(2p-n)\phi_n(x)=0.$$

And (40) shows that

$$\underset{-p}{\overset{p+1}{S}}\,\phi_n^2\frac{1}{(p-x)!\,(p+x)!}=\frac{n!\,(2p)^n}{2^{2n-p}(2p)!},$$

or

$$\underset{-p}{\overset{p}{S}}\,\phi_n^2\frac{2p!\,2^{-2p}}{(p-x)!\,(p+x)!}=\frac{n!(2p)^n}{2^{2n}}.$$

c. *Analogues to Laguerre Polynomials.* Formulas (18), (31), (36). show that the polynomial solutions of the difference equation

$$(x+2h)\underset{h}{\Delta^2}y_n(x)$$
$$+\left[\frac{1-\rho^h}{h}x+2-\rho^h\right]\underset{h}{\Delta}y_n(x)-\frac{1-\rho^h}{h}ny_n(x+h)=0,\ \rho>1,$$

are S-orthogonal on the interval $[0,\infty)$ with the weight function ρ^{-x}; that they are given by the difference coefficient

$$y_n(x)=\rho^x\underset{h}{\Delta}[x^{(nh)}\rho^{-x}];$$

and that they satisfy the recurrent relation

$$y_{n+2}(x)-\left[\frac{\rho^{-h}-1}{h}x+(n+2)\rho^{-h}+(n+1)\right]y_{n+1}(x)+(n+1)^2\rho^{-h}y_n(x)=0.$$

Also, formula (40) yields

$$\overset{\infty}{\underset{0}{S}}\,y_n^2\rho^{-x}\underset{h}{\Delta}x=(n!)^2\frac{h}{1-\rho^{-h}}.$$

Moreover, Theorem 6 states that the zeros of $y_n(x)$ are real, distinct, lie on the interval $[0,\infty)$ and separate those of $y_{n-1}(x)$. The first few polynomials are

$$L_0=1;\quad L_1(x)=\left(\frac{\rho^h-1}{h}\right)\left(x+\frac{\rho^h h}{\rho^h-1}\right);$$
$$L_2(x)=\left(\frac{\rho^h-1}{h}\right)^2\left[x^2+\frac{3h\rho^h-h}{\rho^h-1}x+2\left(\frac{\rho^h h}{\rho^h-1}\right)^2\right];$$
$$L_3(x)=\left(\frac{\rho^h-1}{h}\right)^3\left[x^3+\frac{3h(2\rho^h+1)}{\rho^h-1}x^2+\frac{h^2(11\rho^{2h}+5\rho^h+2)}{(\rho^h-1)^2}x+\frac{6h^3}{(\rho^h-1)^3}\right];\cdots$$

When $\rho=e$ the above polynomials are analogous to the Laguerre Polynomials. In fact, when $y_n(x)$ is multiplied by $1/n!$, if we take the limit as $h\to0$, the above results all reduce to the corresponding relations for the Laguerre Polynomials.

To obtain a simple illustration of these polynomials. let $\rho=1$, $h=1$. then

$$(x+2)\Delta^2y_n(x)-x\Delta y_n(x)+ny_n(x+1)=0$$
$$y_n(x)=2^x\Delta^n(x^{(n)}2^{-x})$$
$$\overset{\infty}{\underset{0}{S}}\,y_n(x)y_m(x)2^{-x}=\begin{cases}0 & (m\neq n)\\ 2^{-n+1}(n!)^2 & (m=n)\end{cases}$$
$$2y_{n+2}(x)-[-x+3n+4]y_{n+1}(x)+(n+1)^2y_n(x)=0.$$

The first few polynomials are

$$P_0 = 1, \quad P_1 = -\tfrac{1}{2}(x-1), \quad P_2 = \tfrac{1}{4}(x^2 - 5x + 2),$$
$$P_3 = -\tfrac{1}{8}(x^3 - 12x^2 + 29x - 6),$$
$$P_4 = \tfrac{1}{16}(x^4 - 22x^3 + 131x^2 - 206x + 24), \cdots$$

d. *Analogue to Jacobi polynomials.* It follows from the general Case I treated above, that the polynomial solutions of the difference equation

$$\begin{aligned}(1 - x - 2h)(1 + x + 2h)&\underset{h}{\Delta^2} y_n(x) \\ &+ [(q-p) - (p+q+2)x + (qp - p - q - 3)h]\underset{h}{\Delta} y_n(x) \\ &+ n(n+1+p+q)y_n(x+h) = 0; \qquad (p > -1, q > -1).\end{aligned}$$

are S-orthogonal on the interval $[-1, 1]$ with the weight function

$$(1+x)^{(qh)}\,(1-x)^{(ph)}.$$

They are given by the difference coefficient

$$y_n(x) = (-1)^n (1+x)^{(-qh)}(1-x)^{(-ph)}\underset{h}{\Delta^n}[(x+1)^{(\overline{q+nh})}(x-1)^{(\overline{p+nh})}]$$

and they satisfy the recurrence relation

$$\begin{aligned}(n+p+q+2)&(2n+p+q+2)y_{n+2}(x) \\ &-(2n+p+q+2)(2n+p+q+3)(2n+p+q+4)xy_{n+1}(x) \\ &+(2n+p+q+3)[p^2 - q^2 - h(p+q)\{(p+1)(q+1) \\ &+ n/2(2p+2q+1)\} + h(2n+4)\{p+q+1 \\ &\qquad\qquad - n/2(p+q-3)\}]y_{n+1}(x) \\ +(n+1)(2n+p+g+4)&[-4\{(n+1)^2 + (n+1)(p+q) + pq\} \\ &+ 2(p-q)\{n(n+p+q+2) + (p+1)(q+1)\}h \\ &+ n(n+p+q+2)\{n + 2(p+1)(q+1)\}h^2]y_n(x) = 0.\end{aligned}$$

11. Limit as $h \to 0$. It is natural to ask, "what happens to the above definitions and properties of the difference equations as $h \to 0$?" We shall show that they reduce to the analogous definitions and properties that have been developed for differential equations.

a. The definition of S-orthogonal functions reduces in the limit, as $h \to 0$, to the integral definition of orthogonal functions, for the sum and the difference quotient reduce to the integral and the derivative respectively.

b. The definition of the L-adjoint difference equation reduces to the definition of the adjoint differential equation. For the difference equation (2) may be written as

$$(2') \qquad q_0(x)\underset{h}{\Delta^{2n}} y(x) + q_1(x)\underset{h}{\Delta^{2n-1}} y(x) + \cdots + q_{2n}(x)y(x) = 0,$$

where

$$q_l(x) = \sum_{j=0}^{l} h^{-l} P_j(x) \frac{(2n-j)(2n-j-1)\cdots(2n-l+1)}{(l-j)!},$$

$$(l = 0, 1, 2, \cdots, 2n)$$

and the L-adjoint equation (5) may be written as

$$(5') \quad \overset{2n}{\underset{h}{\Delta}}(q_0(x-nh)v(x)) - \overset{2n-1}{\underset{h}{\Delta}}[q_1(x-\overline{n-1}h)]v(x+h) + \overset{2n-2}{\underset{h}{\Delta}}[q_2(x-\overline{n-2}h)v(x+2h)] + \cdots + (-1)^{2n} q_{2n}(x+nh)v(x+2nh) = 0;$$

and each of these reduces in the limit, when the limits of the q_l exist, to a differential equation, the second being the adjoint of the first.

The difference equations (1) and (9) approach the differential equation

$$(41) \qquad (ax^2 + bx + c)y''(x) + (dx + f)y'(x) + \lambda y(x) = 0$$

and

$$(42) \qquad q(x)y''(x) + r(x)y'(x) + w(x)y(x) = 0,$$

respectively.

e. The limit as $h \to 0$ of the multiplier (10) is the factor which makes (42) self-adjoint.

Proof.

$$\lim_{h\to 0} t(x) = \lim_{h\to 0} \exp\left(S \frac{1}{h} \log \frac{q(x)}{q(x+h) - hr(x+h)} \underset{h}{\Delta} x \right)$$

$$= \lim_{h\to 0} \exp\left(S \left[\frac{\log q(x) - \log q(x+h)}{h} - \frac{1}{h}\log\left(1 - h\frac{r(x+h)}{q(x+h)}\right)\right] \underset{h}{\Delta} x \right)$$

$$= \lim_{h\to 0} \exp(-\log q(x)) \cdot \exp\left(S \frac{1}{h}\left[h\frac{r(x+h)}{q(x+h)} + h^2\left(\frac{r(x+h)}{q(x+h)}\right)^2 + \cdots \right] \underset{h}{\Delta} x \right), \quad \left| h\frac{r(x+h)}{q(x+h)} \right| < 1.$$

$$= \frac{1}{q(x)} \exp\left(\int \frac{r(x)}{q(x)}\, dx \right).$$

Routh[18] has studied in some detail the polynomial solutions of the differential equation (41). He shows that in general, the polynomial solutions, one for each characteristic value of λ, possess five main properties:

[18] E. J. Routh, "On some properties of certain solutions of a differential equation of the second order," *Proceedings of the London Mathematical Society*, vol. 16 (1884), pp. 245-261. See also, W. C. Brenke, "On polynomial solutions of a class of linear differential equations of the second order," *Bulletin of the American Mathematical Society* (1930), pp. 77-84. Beale, "On the polynomials related to the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} \equiv \frac{N}{D}."$$

Annals of Mathematical Statistics, vol. 8 (1937), pp. 206-23.

1) They may be expressed as a differential coefficient.
2) They satisfy a recurrent relation.
3) They form an orthogonal sequence.
4) Their zeros are all real and confined within certain limits.
5) They possess a generating function.

In the first nine sections of this paper we have shown that the polynomials defined by equation (1) possess three, and in some cases four, of the properties. We have not been able to show the existence of a generating function. We now show that the polynomial solutions of (1) and their properties reduce in the limit, as $h \to 0$, to the polynomial solutions of (41) and their properties that were discovered by Routh.

1) The limit, as $h \to 0$, of the difference form of the polynomial is equal to

$$
\begin{aligned}
&\lim_{h\to 0} \exp\left(S \frac{1}{h} \log \frac{ax^2+bx+c-h(dx+f)}{a(x-h)^2+b(x-h)+c} \underset{h}{\Delta x}\right) \\
&\qquad \times \overset{n}{\underset{h}{\Delta}} \left[\exp\left(-S \frac{1}{h} \log \frac{a(x-nh)^2+(b-dh)(x-nh)+c-hf}{a(x-h)^2+b(x-h)+c} \underset{h}{\Delta x}\right)\right] \\
&\quad = (ax^2+bx+c) \exp\left(\int \frac{dx+f}{ax^2+bx+c}\, dx \right) \\
&\qquad \cdot \frac{d^n}{dx^n}\left[\lim_{h\to 0} \exp\left(S \frac{1}{h} \log \frac{a(x-h)^2+b(x-h)+c}{a(x-nh)^2+b(x-nh)+c} \underset{h}{\Delta x}\right)\right. \\
&\qquad \left. \cdot \exp\left(S \frac{1}{h} \log \left(1-h \frac{d(x-nh)+f}{a(x-nh)^2+b(x-nh)+c}\right) \underset{h}{\Delta x}\right)\right] \\
&\quad = (ax^2+bx+c) \exp\left(\int \frac{dx+f}{ax^2+bx+c}\, dx \right) \\
&\qquad \cdot \frac{d^n}{dx^n}\left[\lim_{h\to 0} \exp\left(\log[a(x-h)^2+b(x-h)+c]^{(\overline{n-1}h)}\right.\right. \\
&\qquad \left.\left. \cdot \exp\left(-\int \frac{dx+f}{ax^2+bx+c}\, dx \right)\right)\right] \\
&\quad = (ax^2+bx+c) \exp\left(\int \frac{dx+f}{ax^2+bx+c}\, dx \right) \\
&\qquad \cdot \frac{d^n}{dx^n}\left[(ax^2+bx+c)^{n-1} \exp\left(\int \frac{dx+f}{ax^2+bx+c}\, dx \right)\right],
\end{aligned}
$$

which is Routh's second differential form for the polynomial solutions of (41).

2) The values of A, B, C, and D in the recurrent relation (34) given by the expression (35), (36), (37), and (38) reduce, respectively, to the four sets of values

$$A = -(n + d/a)(2n + d/a)$$
$$B = (2n + d/a)(2n + 1 + d/a)(2n + 2 + d/a)$$
$$C = -(d/2 - 2)(2n + 1 + d/a)\left(\frac{bd - 2af}{2a^2}\right) + (b/2a)B$$
$$D = (n+1)(2n + 2 + d/a)\left\{\left(\frac{bd - 2af}{2a^2}\right)^2 - (2n + d/a)^2\,\frac{b^2 - 4ac}{4a^2}\right\};$$
$$A = b, \quad B = d, \quad C = 2(n+1)b + f, \quad D = (n+1)\left[\frac{cd - fb}{b} - nb\right];$$
$$A = 1, \quad B = 1, \quad C = f/d, \quad D = -(n+1)c/d;$$
and
$$A = -c, \quad B = d, \quad C = f, \quad D = (n+1)d$$

The first three sets of values are the values obtained by Routh for the three cases he studied.

3). The weight function $t(x - h)$ of the S-orthogonal functions of (1), as shown above, reduce to

$$\frac{1}{ax^2 + bx + c}\exp\left(\int \frac{dx + f}{ax^2 + bx + c}\,dx\right).$$

The interval of S-orthogonality is given by two zeros of

$$[a(x - 2h)^2 + b(x - 2h) + c]$$
$$\times \exp\left(S\frac{1}{h}\log\frac{a(x - 2h)^2 + b(x - 2h) + c}{a(x - h)^2 + b(x - h) + c - h[d(x - h) + f]}\,\Delta_h x\right)$$

and this approaches the function

$$\exp\left(\int \frac{dx + f}{ax^2 + bx + c}\,dx\right),$$

and its zeros give the interval of orthogonality for the polynomials of Routh.

4). In the cases where zero theorems hold, the results reduce to the known results, as $h \to 0$. In other cases, the number of real zeros of the polynomials that lie on the interval of S-orthogonality increase as h decreases.

In particular, the examples (a), (c) and (d), have for their limit their corresponding analogues.

UNIVERSITY OF MARYLAND.

ON AN EXTREMUM PROBLEM IN THE PLANE.*

By Gy. Szekeres.

The following problem was proposed by L. M. Blumenthal [1]: Let M be a set of n arbitrary points in the plane; what is the minimum (as M varies) of the greatest of the $3\binom{n}{3}$ angles formed by the points of M.

If $n = 3, 4, 5, 6$, this minimum is, as stated by Blumenthal $\left(1 - \frac{2}{n}\right)\pi$ attained in the case of the regular polygon. But P. Erdös has shown that there exist configurations of 7 points with every angle $< \left(1 - \frac{1}{3}\right)\pi + \epsilon$ with ϵ arbitrarily small.

In the first part of this paper we show that for $n \geqq 2$ there exist configurations of 2^n points in the plane such that all the angles formed by them are $< \left(1 - \frac{1}{n}\right)\pi + \epsilon$ with ϵ arbitrarily small (Theorem 1).

In the second part we show that among the angles formed by $2^n + 1$ points of the plane there is at least one which is $> \pi\left(1 - \frac{1}{n} + \frac{1}{n(2^n + 1)^2}\right)$, which shows that our results are, in a sense, best possible.

I. Let $A > 2$ be a sufficiently large number. In the plane of the complex numbers we define the set of points $P_0, P_1, P_2, \cdots P_N$, $N = 2^n - 1$,

$$P_0 = 0$$

$$(1)\quad \begin{cases} \text{further if } k = 2^{v_1} + 2^{v_2} + \cdots + 2^{v_m}, \quad 0 \leqq v_1 < v_2 < \cdots < v_m < n \\ \text{then} \qquad P_k = A^{v_1} e^{(v_1/n) i\pi} + A^{v_2} e^{(v_2/n) i\pi} + \cdots + A^{v_m} e^{(v_m/n) i\pi} \end{cases}$$

We prove that every angle formed by any three of these points is $< \left(1 - \frac{1}{n}\right)\pi + \epsilon$ where $\epsilon \to 0$ if $A \to \infty$. Let $k \neq k'$; then

$$P_{k'} - P_k = \delta_{\mu_1} A^{\mu_1} e^{(\mu_1/n) i\pi} + \cdots + \delta_{\mu_j} A^{\mu_j} e^{(\mu_j/n) i\pi}$$

where

$$\mu_1 < \cdots < \mu_j, \qquad \delta_\mu = \pm 1$$

and μ_j is the greatest exponent occurring in only one of the representations (1) of k and k'. Hence

* Received May 6, 1940; revised October 9, 1940.

[1] L. M. Blumenthal, "Metric methods in determinant theory," *American Journal of Mathematics*, vol. 61 (1939), pp. 912-922.

$$\frac{P_{k'} - P_k}{A^{\mu j}} = \delta_{\mu_j} e^{(\mu j/n)\pi i} + \eta$$

where

$$|\eta| < \frac{1}{A}\left(1 + \frac{1}{A} + \cdots\right) < \frac{2}{A}$$

Thus the direction of $P_{k'} - P_k$ is arbitrarily near to $\pm e^{(\mu j/n)\pi i}$ if only A is sufficiently large. We shall call $\delta_{\mu_j} e^{(\mu j/n)\pi i}$ the approximate direction of $P_{k'} - P_k$; in symbols

$$D(k' - k) = \delta_{\mu_j} e^{(\mu j/n)\pi i}.$$

Now let $k' \neq k''$, and

$$(2) \qquad \begin{cases} D(k' - k) = \delta e^{(\mu/n)\pi i} \\ D(k'' - k) = \delta' e^{(\mu/n)\pi i} \\ D(k'' - k') = \delta'' e^{(\mu'/n)\pi i} \end{cases}$$

Then $\mu' \neq \mu$ and $\delta = \delta'$; for, if the representation (1) of k contains the exponent μ, then that of k' and k'' cannot contain it; on the other hand, if μ is not contained in the representation (1) of k, the k' and k'' must both contain it. In either case we have $\mu' \neq \mu$.

Now suppose $\delta = -\delta'$; then the vectors $P_{k'} - P_k$ and $P_{k''} - P_k$ are in (approximately) opposite direction, and we would have $D(k'' - k') = D(k'' - k)$ against our former assertion. Hence $\delta = \delta'$ in (2).

But obviously the approximate angle formed by any two $P_{k'} - P_k$ and $P_{k''} - P_k$ is $1 - j/n$, $j = 0, 1, \cdots n$ and here the case $j = 0$ is excluded by the above considerations. This completes the proof of Theorem 1. To prove the second half of our theorem we put forward some considerations in the theory of graphs. We call a system of points $Q_1, Q_2 \cdots$ and edges Q_iQ_j connecting these points a graph. We call a system of edges $Q_1Q_2, Q_2Q_3 \cdots Q_{i-1}Q_i$ a path. If $Q_1 = Q_i$ we have a closed path. If every two of the points Q_i and Q_j are connected the graph is called complete. The proof of Theorem 2 is based on the following

LEMMA: *Let G be a complete graph of $N > 2^n$ points; then it cannot be the union of n graphs $g_1, g_2, \cdots g_n$ such that in every g all closed paths contain an even number of edges.*

Proof. We use complete induction. The Lemma evidently holds for $n = 1$. Assume that it is true for $n - 1$ and that

$$G = g_1 + g_2 + \cdots g_n$$

where in each g_i all the closed paths have an even number of vertices. It is well known [2] that we can divide the vertices of g_1 into two classes A and B such

[2] Dénes König, *Theorie der endlichen und unendlichen Graphen*, p. 170.

that every edge of g_1 connects a point of A with a point of B. Hence we can also divide the vertices of G into two disjoint classes A' and B' such that every edge of g_1 connects a point of A' with a point of B'. We can assume that the number of points of A' is not less than $\frac{N}{2} > 2^{n-1}$. But then

$$g_2 + g_3 + \cdots g_n$$

would contain a complete graph (consisting of all joins of pairs of points of A') the number of points of which is greater than 2^{n-1}, which contradicts the induction hypothesis; this completes the proof.

Now we prove Theorem 2. Let $N > 2^n$ points $P_1, P_2, \cdots P_n$ be given in the plane. Connect any two of them; there evidently exists a line O such that all the angles formed by O and any of the lines P_iP_j are numerically greater than $\frac{\pi}{N^2}$. Denote by $K_r^{(1)}$ the set of oriented lines in the plane whose angle with O taken positively lies between

$$\frac{\pi}{N^2} + (r-1)\,\frac{\pi - \pi/N^2}{n} \text{ and } \frac{\pi}{N^2} + r\,\frac{\pi - \pi/N^2}{n} \text{ (inclusive)}, \quad r = 1, 2, \cdots n.$$

and similarly by $K_r^{(2)}$ the set of lines whose angle with O lies between

$$\pi\left(1 + \frac{1}{N^2}\right) + (r-1)\frac{\pi(1 - 1/N^2)}{n} \text{ and } \pi\left(1 + \frac{1}{N^2}\right) + r\frac{\pi(1 - 1/N^2)}{n}$$

(inclusive).

Denote now by g_i the graph formed by the lines P_iP_j directed from P_i to P_j in $(K_r^{(1)} + K_r^{(2)})$. Evidently the union of $g_1, g_2, \cdots g_n$ contains the complete graph formed by our N points; thus by our Lemma at least one of the graphs g say g_i contains a closed path with an odd number of vertices P_1, $P_2, \cdots P_{2r+1}$. But then there exist three consecutive vertices, say P_{l-1}, P_l, P_{l+1} such that both lines $P_{l-1}P_l$ and P_lP_{l+1} lie either in $K_r^{(1)}$ or in $K_r^{(2)}$. In either case the angle $P_{l-1}P_lP_{l+1}$ is not less than $\pi\left(1 - \frac{1}{n} + \frac{1}{nN^2}\right)$ which completes the proof.

We can ask similarly the minimum of the greatest of the angles formed by N points in three dimensional space. Our method does not given the exact answer in this case. We can only prove that, on the one hand among $2^n + 1$ points there are always three which form an angle $> \left(1 - \frac{c_1}{\sqrt{n}}\right)\pi$; on the other hand there exist 2^n points such that the maximum angle is $< \left(1 - \frac{c_2}{\sqrt{n}}\right)\pi$ where c_1 and c_2 are constants.

59 Brenan Road, Shanghai.

EXPLICIT BOUNDS FOR SOME FUNCTIONS OF PRIME NUMBERS.*

By Barkley Rosser.

Summary of results. Counting 2 as the first prime, we denote by $\pi(x)$, $p(n)$, and $\theta(x)$, respectively, the number of primes less than or equal to x, the n-th prime, and the logarithm of the product of all primes less than or equal to x. It is known that for each positive constant A, there is a constant N for which the three following statements hold true.

If $N \leq x$, then

$$\frac{x}{\log x - 1 + A} < \pi(x) < \frac{x}{\log x - 1 - A}.$$

If $N \leq n$, then

$$n \log n + n \log \log n - n - An < p(n) < n \log n + n \log \log n - n + An.$$

If $N \leq n$, then

$$\left(1 - \frac{A}{\log x}\right) x < \theta(x) < \left(1 + \frac{A}{\log x}\right) x.$$

Moreover, these three statements are essentially interdeducible, in the sense that if one of them can be proved for a certain value, A^*, of A, then the other two can be proved with the value $A^* + \epsilon$ for A, where ϵ depends on N.

Heretofore the question of determining the N which goes with a particular A has received no attention. Moreover, the question of how small A can be taken without requiring that N become large has been neglected.

Theorem 22 of this paper furnishes an explicit answer (not the best possible) to the first question. For the second question the answer $A = 3$ is given. Again this is not the best possible, and a partial proof is given that we can take $A = 1$. In particular, for $A = 3$, we have:

Theorem 29. *If* $55 \leq x$, *then:*

A. $$\frac{x}{\log x + 2} < \pi(x) < \frac{x}{\log x - 4}.$$

B. $$x \log x + x \log \log x - 4x < p([x]) < x \log x + x \log \log x + 2x.$$

C. $$\left(1 - \frac{3}{\log x}\right) x < \theta(x) < \left(1 + \frac{3}{\log x}\right) x.$$

* Received June 28, 1940.

For $A = 1$, we have the partial result embodied in the following seven theorems.

THEOREM 30. *If $e^{2000} \leq x$, then:*

A. $\dfrac{x}{\log x} < \pi(x) < \dfrac{x}{\log x - 2}$.

B. $x \log x + x \log\log x - 2x < p([x]) < x \log x + x \log\log x$.

C. $\left(1 - \dfrac{1}{\log x}\right) x < \theta(x) < \left(1 + \dfrac{1}{\log x}\right) x$.

THEOREM 26. *If $17 \leq x \leq e^{100}$, then* $\dfrac{x}{\log x} < \pi(x)$.

THEOREM 25. *If $e^2 < x \leq e^{100}$, then* $\pi(x) < \dfrac{x}{\log x - 2}$.

THEOREM 27. *If $1 < n \leq e^{100}$, then* $n \log n + n \log\log n - 2n < p(n)$.

THEOREM 28. *If $6 \leq n \leq e^{95}$, then* $p(n) < n \log n + n \log\log n$.

THEOREM 23. *If $41 \leq x \leq e^{100}$, then* $\left(1 - \dfrac{1}{\log x}\right) x < \theta(x)$.

THEOREM 24. *If $1 < x \leq e^{100}$, then* $\theta(x) < \left(1 + \dfrac{1}{\log x}\right) x$.

For $0 < x \leq 1{,}000{,}000$, these results were derived from the very sharp theorem:

If $0 < x \leq 1{,}000{,}000$, then $x - 2.78x^{\frac{1}{2}} < \theta(x) < x$;

which was proved essentially by comparison with Lehmer's "List of Prime Numbers." For $1{,}000{,}000 \leq x$, analytical methods were used. It is remarkable that in the ranges $1{,}000{,}000 \leq x \leq e^{95}$ and $e^{2000} \leq x$, the analytical methods enable one to take $A = 1$, whereas in the intermediary range $e^{95} \leq x \leq e^{2000}$ one has to be satisfied with a larger value of A. This peculiar situation is apparently due to the insufficiency of our present knowledge about the zeros of the Riemann zeta function. A significant increase in our present knowledge would undoubtedly enable one to take $A = 1$ in the intermediary range also. For instance, if $\beta + i\gamma$ is a typical complex zero of the zeta function, it would suffice to know that

$$\beta < 1 - \frac{1}{6 \log \gamma}$$

for $1400 \leq \gamma$.

References to the bibliography will consist of a number referring to the numbered bibliography at the end of the paper followed by the necessary page references.

Tables of $\psi(x)$, $\theta(x)$, and $p(n)$. We define

$$\psi(x) = \sum_{p^m \leq x} \log p.$$

The following relations connect $\theta(x)$ and $\psi(x)$:

$$(1) \qquad \psi(x) = \sum_{n=1}^{[x]} \theta(x^{1/n}) = \theta(x) + \theta(x^{\frac{1}{2}}) + \theta(x^{1/3}) + \cdots$$

$$\theta(x) = \sum_{n=1}^{[x]} \mu(n)\psi(x^{1/n})$$

$$(2) \qquad = \psi(x) - \psi(x^{1/2}) - \psi(x^{1/3}) - \psi(x^{1/5}) + \psi(x^{1/6}) - \cdots.$$

Gram has given a table of $\psi(x)$ to eight decimal places for $1 \leq x \leq 2000$ (see 1, pp. 281-288). As a check on Gram's table, the present author computed $\psi(x)$ for $1 \leq x \leq 2000$ by use of a six place table of natural logarithms and found no discrepancies outside the limits of accuracy of the computation. By comparing the first few entries of Gram's table with values computed from eight place, nine place, ten place, and eleven place values of the natural logarithms of the primes, it was apparent that Gram must have used eight place values in computing his table. Hence the eighth place in Gram's table is not reliable.

By use of (2) and Gram's table, one can readily compute $\theta(x)$ for $1 \leq x \leq 2000$. In order to facilitate the computation of $\theta(x)$ for $x \leq 10{,}000$, Table I was prepared (see end of paper). Table I is auxiliary to a table of Jones (2, pp. 114-117. Note two errors: $\log 6899 = 8.839132$ and $\log 7853 = 8.968651$). Jones's table gives the natural logarithms to six decimal places of the primes from 2000 to 10,000. To compute $\theta(x)$, one can take the nearest entry below x in Table I, and add the logarithms of the intervening primes as given in Jones's table. As a check, one can take the next entry above x, and subtract the logarithms of the intervening primes. So that this check will come out exact, Table I was computed from Jones's table, and all six places were retained, though the last is quite unreliable. Table I was checked by adding up seven place common logarithms of the primes, and multiplying the sums by $\log_e 10$.

Lehmer's List of Prime Numbers (3, pp. 1-135) is a tabulation of $p(n-1)$ as a function of n for $1 \leq n \leq 675{,}000$ (Lehmer takes 1 as the first prime). From this table, one can read off $\pi(x) + 1$ for $1 \leq x \leq 10{,}006{,}721$.

Relations between $\pi(x)$, $p(n)$, and $\theta(x)$. Obvious relations are

$$\pi(p(n)) = n \tag{3}$$

$$p(\pi(x)) \leq x \tag{4}$$

$$\theta(p(n)) = \sum_{r=1}^{n} \log p(r). \tag{5}$$

By use of a theorem connecting sums with integrals (4, Theorem A, p. 18), we get

$$\theta(x) = \pi(x) \log x - \int_2^x \frac{\pi(y)\,dy}{y} \tag{6}$$

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(y)\,dy}{y \log^2 y}. \tag{7}$$

From these we now proceed to deduce relations which hold between functions which bound $\pi(x)$, $p(n)$, and $\theta(x)$. In constructing bounding functions for $\pi(x)$, the logarithmic integral,

$$li(x) = \lim_{\eta \to +0} \left\{ \int_0^{1-\eta} \frac{dy}{\log y} + \int_{1+\eta}^{x} \frac{dy}{\log y} \right\},$$

is very convenient to use, so we call attention to a few tables of $li(x)$ and the related function

$$Ei(x) = li(e^x).$$

The most complete table of $Ei(x)$ is Table VII in Vol. 1 of the Mathematical Tables of the British Association for the Advancement of Science. A short table of $Ei(x)$ is given in Chapter VI of Funktionentafeln, by Jahnke and Emde. Lehmer gives a table of $li(x)$ correct to the nearest integer (3, pp. xiii-xvi. The column headed "Tchebycheff" contains vaules of $li(x)$, and not values of $\int_2^x \frac{dy}{\log y}$ as Lehmer claims).

LEMMA 1. $\displaystyle\int_2^{e^{2.4}} \frac{\theta(y)\,dy}{y \log^2 y} < li(e^{2.4}) - \frac{e^{2.4}}{2.4}$.

Proof. The expression on the right has the value 2.008, whereas, by (7), the expression on the left has the value 1.773.

A similar proof using (6) holds for the next two lemmas.

LEMMA 2. $\displaystyle\int_2^{e^4} \frac{\pi(y)\,dy}{y} > 4li(e^4) - e^4 + 2e^2 - 4li(e^2)$.

LEMMA 3. $\displaystyle\int_2^{e^3} \frac{\pi(y)\,dy}{y} < 3li(e^3) - e^3$.

LEMMA 4. *If* $\theta(x) < x$ *for* $e^{2.4} \leq x \leq K$, *then* $\pi(x) < li(x)$ *for* $e^{2.4} \leq x \leq K$.

Proof. Under the hypothesis of the theorem we can deduce from (7) and Lemma 1 that, if $e^{2.4} \leq x \leq K$, then

$$\pi(x) < \frac{x}{\log x} + li(e^{2.4}) - \frac{e^{2.4}}{2.4} + \int_{e^{2.4}}^{x} \frac{y\,dy}{y \log^2 y}.$$

However, integration by parts gives

$$\int \frac{y\,dy}{y \log^2 y} = -\frac{y}{\log y} + \int \frac{dy}{\log y},$$

from which the lemma follows.

For the next two lemmas, the proof is similar and makes use of the fact that integration by parts gives

$$\int \frac{li(y)dy}{y} = \log y\, li(y) - y,$$

and that if $y = z^2$

$$\int \frac{li(y^{\frac{1}{2}})dy}{y} = 2 \int \frac{li(z)dz}{z}.$$

LEMMA 5. *If* $li(x) - li(x^{\frac{1}{2}}) < \pi(x)$ *for* $e^4 \leq x \leq K$, *then*

$$\theta(x) < x - 2x^{\frac{1}{2}} + \log x(\pi(x) - li(x) + li(x^{\frac{1}{2}}))$$

for $e^4 \leq x \leq K$.

LEMMA 6. *If* $\pi(x) < li(x)$ *for* $e^3 \leq x \leq K$, *then*

$$x - \log x(li(x) - \pi(x)) < \theta(x)$$

for $e^3 \leq x \leq K$.

COROLLARY. *If* $li(x) - li(x^{\frac{1}{2}}) < \pi(x) < li(x)$ *for* $e^3 \leq x \leq K$, *then* $x - \log x\, li(x^{\frac{1}{2}}) < \theta(x)$ *for* $e^3 \leq x \leq K$.

LEMMA 7. *For* $e^4 \leq x$, $li(x) < \dfrac{x}{\log x - 2}$.

Proof. For $x = e^4$, the inequality is true, and for $e^4 \leq x$,

$$\frac{d}{dx} li(x) \leq \frac{d}{dx} \frac{x}{\log x - 2}.$$

LEMMA 8. *For* $e^4 \leq x$, $\dfrac{x}{\log x} < li(x) - li(x^{\frac{1}{2}})$.

Similar proof.

LEMMA 9. *For* $n \geq 5$, $\theta(p(n)) > n \log n + n \log\log n - n - li(n)$.

Proof. Since $p(r) > r \log r$ (5, Theorem 1, p. 37), we have by (5),

$$\begin{aligned}
\theta(p(n)) &\geq \theta(p(5)) + \sum_{r=6}^{n} \log(r \log r) \\
&\geq \theta(p(5)) + \int_5^n \log x\, dx + \int_5^n \log\log x\, dx \\
&\geq \theta(p(5)) + n \log n - n - 5 \log 5 + 5 \\
&\quad + n \log\log n - 5 \log\log 5 + li(5) - li(n) \\
&> n \log n + n \log\log n - n - li(n).
\end{aligned}$$

LEMMA 10. *If* $16 \leq n$ *and if* $p(r) < r \log r + r \log\log r$ *for* $16 \leq r < n$, *then*

$$\theta(p(n)) < n \log n + n \log\log n - n + \frac{n \log\log n}{\log n}.$$

Proof. Since $p(n) < n \log n + 2n \log\log n$ for $n \geq 3$ (5, Theorem 2, p. 40), we have

$$\begin{aligned}
\theta(p(n)) &< \theta(p(15)) + \sum_{r=16}^{n-1} \log(r \log r + r \log\log r) + \log(n \log n + 2n \log\log n) \\
&< \theta(p(15)) + \int_{16}^n \log x\, dx + \int_{16}^n \log(\log x + \log\log x)\, dx \\
&\quad + \log(n \log n + 2n \log\log n) \\
&< \theta(p(15)) + n \log n - n - 16 \log 16 + 16 + n \log(\log n + \log\log n) \\
&\quad - 16 \log(\log 16 + \log\log 16) - \int_{16}^n \frac{(\log x + 1)\, dx}{\log x(\log x + \log\log x)} \\
&\quad + \log(n \log n + 2n \log\log n).
\end{aligned}$$

However

$$\log(\log n + \log\log n) = \log\log n + \log\left(1 + \frac{\log\log n}{\log n}\right) < \log\log n + \frac{\log\log n}{\log n},$$

and

$$\int_{16}^n \frac{(\log x + 1)\, dx}{\log x(\log x + \log\log x)} > \int_{16}^n \frac{dx}{2 \log x},$$

and

$$\begin{aligned}
&\theta(p(15)) - 16 \log 16 + 16 - 16 \log(\log 16 + \log\log 16) \\
&\qquad - \tfrac{1}{2}(li(n) - li(16)) + \log(n \log n + 2n \log\log n) < 0.
\end{aligned}$$

The range $0 < x \leq 1{,}000{,}000$.

THEOREM 1. *For* $11 \leq x \leq 1{,}000{,}000$, $li(x) - li(x^{\frac{1}{2}}) < \pi(x)$.

Proof. The interval $0 < x \leq 1{,}000{,}000$ was divided into convenient sub-

intervals. Corresponding to each subinterval, I_r, a linear function, $Ax + B$, was determined so that

$$li(x) - li(x^{\frac{1}{2}}) \leq Ax + B$$

for x in I_r. Then by comparison with Lehmer's List of Prime Numbers, it was determined that $Ax + B + 1 < \pi(x) + 1$ for x in I_r. To show how this can be done quickly, we exhibit a specimen of the computation. By getting the equation of the tangent to the curve

$$y = li(x) - li(x^{\frac{1}{2}})$$

at $x = e^{13.4}$, and by noting that the curve is convex upward, we ascertain that

$$li(x) - li(x^{\frac{1}{2}}) < 4{,}323.1 + \frac{x}{13.41651}$$

for $x > 2$. So we undertake to show that

$$4{,}324.1 + \frac{x}{13.41651} < \pi(x) + 1$$

in I_r. Put $x = e^{13.4} + 13.41651\, y = a + by$. Then it suffices to show that

$$53{,}518 + y \leq \pi(a + by) + 1 \tag{8}$$

for $a + by$ in I_r. Now, using a Monroe High Speed Adding Calculator (or any computing machine of similar construction), put 53,518 on the upper dials, put a (that is, 660,003.22477) on the lower dials, and b (that is, 13.41651) on the keyboard. Now if one holds the + bar down for y revolutions, $53{,}518 + y$ will appear on the upper dials and $a + by$ will appear on the lower dials. In other words, we have now set the machine so that it will readily give $a + by$ as a function of $53{,}518 + y$. Now take $53{,}518 + y = 53{,}550$ and compute $a + by$. Then we see by the list of primes that

$$p(53{,}518 + y + 50 - 1) < a + by,$$

so that

$$53{,}518 + y + 50 < \pi(a + by) + 1,$$

and so (8) holds for $53{,}550 \leq 53{,}518 + y \leq 53{,}600$. Put $53{,}518 + y = 53{,}600$, and then

$$p(53{,}518 + y + 50 - 1) < a + by,$$

so that (8) holds for $53{,}600 \leq 53{,}518 + y \leq 53{,}650$. And so on. In general, for x in the neighborhood of $e^{13.4}$, one can advance y by 50 at a time. Of course irregularities in the distribution of the primes cause trouble occasionally. For instance, if $53{,}518 + y = 53{,}800$ then $\pi(a + by) + 1 = 53{,}848$, so it would appear that one could only advance y by 48. However

$p(53{,}850 - 1)$ only exceeds $a + by$ by eleven, so that one can readily see that if one increased y by 48, then $a + by$ would be enough larger to justify advancing y by 2 more.

For x in other neighborhoods, one would advance y uniformly by some amount different from 50. Three factors determine the choice of this amount:

I. It should be small enough so that exceptional cases occur infrequently.

II. For speed in operating the machine, it should be a multiple of 10 if possible.

III. For convenience in locating entries in the list of primes, it should be a factor or multiple of 100.

THEOREM 2. *For* $0 < x \leq 1{,}000{,}000$, $\theta(x) < x$.

Proof. Clearly it suffices to prove $\theta(p(n)) < p(n)$ for $p(n) \leq 1{,}000{,}000$. As in the proof of Theorem 1, we divide the interval $0 < x \leq 1{,}000{,}000$ into subintervals. So suppose we wish to prove $\theta(p(n+h)) < p(n+h)$ for $n \leq n + h \leq N$. If $p(n) \leq 10{,}000$, compute $\theta(p(n))$. If $p(n) > 10{,}000$, use Theorem 1 and Lemma 5 to compute a k such that $\theta(p(n)) < k$. Then

$$\theta(p(n+h)) < k + h \log N.$$

So it suffices to prove that

$$k + h \log N < p(n+h).$$

That is, it suffices to prove that

$$\pi(k + h \log N) + 1 < n + 1 + h.$$

Put $x = k + h \log N$, and we have the problem reduced to that of comparing $\pi(x) + 1$ with $Ax + B + 1$ in a given range, and appropriate modifications of the technique described in the proof of Theorem 1 will work.

COROLLARY. *For* $1 < x \leq 1{,}000{,}000$, $\theta(x) < \left(1 + \dfrac{1}{\log x}\right) x$.

THEOREM 3. *For* $2 \leq x \leq 1{,}000{,}000$, $\pi(x) < li(x)$.

Proof. For $2 \leq x \leq e^{2.4}$, compare values of $\pi(x)$ and $li(x)$. For $e^{2.4} \leq x \leq 1{,}000{,}000$ use Theorem 2 and Lemma 4.

THEOREM 4. *For* $7 \leq x \leq 1{,}000{,}000$, $x - \log x \, li(x^{\frac{1}{2}}) < \theta(x)$.

Proof. For $7 \leq x \leq e^{3}$, compare values. For $e^{3} \leq x \leq 1{,}000{,}000$, use Lemma 6, Cor.

Remarks. If $10{,}000 < x \leq 1{,}000{,}000$, and one wishes more exact bounds for $\theta(x)$ than those given in Theorem 2 and Theorem 4, they can be obtained from the List of Prime Numbers with the aid of Lemma 5 and Lemma 6.

It seems likely that Theorems 1-4 remain true if the upper limit of 1,000,000 is replaced by an upper limit of 10,000,000. However it is known that all four theorems are false if one tries to replace the upper limit by infinity (4, Chapter V, especially Theorem 34 and Theorem 35).

THEOREM 5. *For* $0 < x \leq 1420$ *and for* $1423 \leq x \leq 10{,}000$,

$$x - 2x^{\frac{1}{2}} < \theta(x).$$

Proof. Compare values.

Remark. For $x = 1421$ and $x = 1422$, $\theta(x) < x - 2x^{\frac{1}{2}}$.

THEOREM 6. *For* $0 < x \leq 10{,}000$, $x - 2.025807x^{\frac{1}{2}} < \theta(x)$.

THEOREM 7. *For* $0 < x \leq 1{,}000{,}000$, $x - 2.78x^{\frac{1}{2}} < \theta(x)$.

Proof. By Theorem 4 and Theorem 6, one only needs to prove that $\log x \, li(x^{\frac{1}{2}}) \leq 2.78x^{\frac{1}{2}}$ for $10{,}000 \leq x$, and this is not difficult.

A slight additional computation allows us to infer:

COROLLARY. *For* $41 \leq x \leq 1{,}000{,}000$, $\left(1 - \dfrac{1}{\log x}\right) x < \theta(x)$.

THEOREM 8. *For* $17 \leq x \leq 1{,}000{,}000$, $\dfrac{x}{\log x} < \pi(x)$.

Proof. For $17 \leq x \leq e^4$, compare values. For $e^4 \leq x \leq 1{,}000{,}000$, use Theorem 1 and Lemma 8.

THEOREM 9. *For* $e^2 < x \leq 1{,}000{,}000$, $\pi(x) < \dfrac{x}{\log x - 2}$.

Similar proof using Theorem 3 and Lemma 7.

THEOREM 10. *For* $1 < x \leq 83{,}498$, $n \log n + n \log \log n - 2n < p(n)$.

Proof. For $1 < n \leq 1480$, $\log \log n < 2$, so that the theorem follows from the fact that $p(n) > n \log n$ (5, Theorem 1, p. 37). For $1480 \leq n \leq 83{,}498$, $\theta(p(n)) < p(n)$ by Theorem 2 and $li(n) < n$, so that the result follows by Lemma 9.

THEOREM 11. *If* $6 \leq n \leq 83{,}498$, $p(n) < n \log n + n \log \log n$.

Proof. For $6 \leq n \leq e^5$, compare values. For $e^5 \leq n \leq 83{,}498$, we use induction on n. Assume $p(r) < r \log r + r \log \log r$ for $6 \leq r < n$. Then

we wish to prove that $p(n) < n \log n + n \log\log n$. So suppose $p(n) \geq n \log n + n \log\log n$. Then

$$\theta(n \log n + n \log\log n) \leq \theta(p(n)).$$

So by Theorem 7 and Lemma 10,

$$n \log n + n \log\log n - 2.78(n \log n + n \log\log n)^{\frac{1}{2}}$$
$$< n \log n + n \log\log n - n + \frac{n \log\log n}{\log n}.$$

That is,

$$n < 2.78(n \log n + n \log\log n)^{\frac{1}{2}} + \frac{n \log\log n}{\log n},$$

which is false if $n \geq e^5$.

Bounds for $\psi(x)$. Define

$$\phi(x) = \psi(x) - x + \log 2\pi + \tfrac{1}{2}\log(1 - 1/x^2).$$

$$K_m(x, h) = \int_0^h dy_1 \int_0^h dy_2 \cdots \int_0^h \phi(x + y_1 + y_2 + \cdots + y_m)\, dy_m.$$

$$f_{m,n,a}(x, h, z) = \frac{K_m(x, h)}{h^n} + \tfrac{1}{2} n h^a - z h^{a-1}.$$

In the next five lemmas we assume $x > 1$ and $x + mh > 1$ so that the functions and integrals discussed will all exist and take only real values.

LEMMA 11. *Obviously*

$$\int_0^h f_{m,1,a}(x, h, z)\, dz = K_m(x, h).$$

LEMMA 12. *Obviously*

$$\int_0^h f_{m,n,a}(x, h, y_1 + y_2 + \cdots + y_n)\, dy_n = f_{m,n-1,a+1}(x, h, y_1 + y_2 + \cdots + y_{n-1}).$$

LEMMA 13. $K_m(x, h)$

$$= \int_0^h dy_1 \int_0^h dy_2 \cdots \int_0^h f_{m,n,a}(x, h, y_1 + y_2 + \cdots + y_n)\, dy_n.$$

Proof by induction on n, using Lemmas 11-12.

Define

$$f_m(x, h, z) = f_{m,m,1}(x, h, z).$$

LEMMA 14. *If* $0 < h$, *then there is a* z *such that* $0 \leq z \leq mh$ *and*

$$\phi(x + z) \leq f_m(x, h, z).$$

Proof. Suppose that $\phi(x + z) > f_m(x, h, z)$ for $0 \leq z \leq mh$. Then

$$\int_0^h dy_1 \int_0^h dy_2 \cdots \int_0^h \phi(x + y_1 + y_2 + \cdots + y_m)\,dy_m$$
$$> \int_0^h dy_1 \int_0^h dy_2 \cdots \int_0^h f_m(x, h, y_1 + y_2 + \cdots + y_m)\,dy_m.$$

However the former equals $K_m(x, h)$ by definition, and the latter equals $K_m(x, h)$ by Lemma 13.

LEMMA 15. *If $h < 0$, then there is a z such that $mh \leq z \leq 0$ and*

$$\phi(x + z) \geq f_m(x, h, z).$$

Proof. Suppose that $\phi(x + z) < f_m(x, h, z)$ for $mh \leq z \leq 0$. Then

$$\int_h^0 dy_1 \int_h^0 dy_2 \cdots \int_h^0 \phi(x + y_1 + y_2 + \cdots + y_m)\,dy_m$$
$$< \int_h^0 dy_1 \int_h^0 dy_2 \cdots \int_h^0 f_m(x, h, y_1 + y_2 + \cdots + y_m)\,dy_m.$$

However the former equals $(-1)^m K_m(x, h)$ by definition, and the latter equals $(-1)^m K_m(x, h)$ by Lemma 13.

THEOREM 12. *If $0 < \delta < (x - 1)/xm$, and*

$$\epsilon_1 = \frac{K_m(x, -x\delta)}{(-x)^{m+1}\delta^m} + \frac{m\delta}{2}, \qquad \epsilon_2 = \frac{K_m(x, x\delta)}{x^{m+1}\delta^m} + \frac{m\delta}{2},$$

then

$$x(1 - \epsilon_1) - \log 2\pi - \tfrac{1}{2}\log(1 - 1/x^2) \leq \psi(x)$$
$$\leq x(1 + \epsilon_2) - \log 2\pi - \tfrac{1}{2}\log(1 - 1/x^2).$$

Proof. Put $h = \delta x$. Use Lemma 14, and put in the definitions of $\phi(x)$ and $f_m(x, h, z)$. Hence there is a z such that $0 \leq z \leq mh$, and

$$\psi(x + z) - (x + z) + \log 2\pi + \tfrac{1}{2}\log\left(1 - \frac{1}{(x + z)^2}\right) \leq \frac{K_m(x, h)}{h^m} + \frac{mh}{2} - z.$$

Replace h by $x\delta$, and one has

$$\psi(x + z) \leq x(1 + \epsilon_2) - \log 2\pi - \tfrac{1}{2}\log\left(1 - \frac{1}{(x + z)^2}\right).$$

However, $0 \leq z$, so that $\psi(x) \leq \psi(x + z)$ and

$$-\tfrac{1}{2}\log\left(1 - \frac{1}{(x + z)^2}\right) \leq -\tfrac{1}{2}\log(1 - 1/x^2).$$

This proves half the theorem. To prove the other half, put $h = -x\delta$, and proceed similarly, using Lemma 15.

Henceforth, we denote non-trivial zeros (4, p. 58) of $\zeta(s)$ by $\rho = \beta + i\gamma$.

THEOREM 13. $\int_0^h \phi(x+z)dz = \sum_\rho \frac{1}{\rho(\rho+1)} \{x^{\rho+1} - (x+h)^{\rho+1}\}.$

Proof. It is known (6, p. 317) that

$$\frac{\zeta'(0)}{\zeta(0)} = \log 2\pi,$$

and also (4, p. 30, p. 73) that

$$\int_1^x \psi(u)du = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - x\frac{\zeta'(0)}{\zeta(0)} + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{r=1}^{\infty} \frac{x^{1-2r}}{2r(2r-1)}.$$

The theorem follows from these and the fact that $\sum_\rho |\rho|^{-2}$ is convergent (4, Theorem 18, p. 57). This latter fact enables us to integrate the result of Theorem 13 term by term and deduce Theorem 14.

THEOREM 14. $K_m(x, \pm x\delta) =$

$$\sum_\rho \frac{x^{\rho+m}}{\rho(\rho+1)\cdots(\rho+m)} \{\sum_{j=0}^{m} (-1)^{j+m+1} \binom{m}{j} (1 \pm j\delta)^{\rho+m}\}.$$

Define

$$K = K(m, x) = \sum_\rho \frac{x^{\beta-1}}{|\gamma^{m+1}|}. \tag{9}$$

THEOREM 15. *If* $0 \leq \delta$, *then*

$$|K_m(x, \pm x\delta)| < x^{m+1}((1+\delta)^{m+1}+1)^m K.$$

Proof. Take absolute values of both sides of Theorem 14.

$$\left|\frac{x^{\rho+m}}{\rho(\rho+1)\cdots(\rho+m)}\right| < \frac{x^{\beta+m}}{|\gamma^{m+1}|}$$

(4, Theorem 16, p. 48). Also

$$|(1 \pm j\delta)^{\rho+m}| = (1 \pm j\delta)^{\beta+m} < (1+j\delta)^{m+1}.$$

So

$$\begin{aligned}
\left|\sum_{j=0}^{m} (-1)^{j+m+1} \binom{m}{j} (1 \pm j\delta)^{\rho+m}\right| &< \sum_{j=0}^{m} \binom{m}{j} (1+j\delta)^{m+1} \\
&< \sum_{j=0}^{m} \binom{m}{j} ((1+\delta)^j)^{m+1} \\
&= \sum_{j=0}^{m} \binom{m}{j} ((1+\delta)^{m+1})^j \\
&= ((1+\delta)^{m+1}+1)^m.
\end{aligned}$$

THEOREM 16. *With* K *as in* (9) *and* ϵ_1 *and* ϵ_2 *as in Theorem* 12, *if one takes*

$$\delta \geq 2K^{1/(m+1)}, \qquad \theta = \frac{\delta}{2}\left\{\left(\frac{(1+\delta)^{m+1}+1}{2}\right)^m + m\right\},$$

then $\epsilon_1 < \theta$ and $\epsilon_2 < \theta$.

Proof by use of Theorem 15.

We now derive an upper bound for K. First we need information about the zeros of $\zeta(s)$. Let $N(T)$ denote the number of ρ's for which $0 < \gamma \leq T$. Define

$$F(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8},$$

$$R(T) = 0.137 \log T + 0.443 \log\log T + 1.588.$$

THEOREM 17. *For* $0 \leq T \leq 280$, $|N(T) - F(T)| < 1$.

Proof. This can be deduced from the computations of Hutchinson (7, pp. 49-60).

Choose A so that $F(A) = 1041$. Then $A = 1467.47747$ correct to five decimal places.

THEOREM 18. *For* $0 < T \leq A$, $|N(T) - F(T)| < 2$; $N(A) = F(A) = 1041$; *and for* $0 < \gamma \leq A$, $\beta = \frac{1}{2}$.

Proof. The computations of Titchmarsh (8, pp. 234-250, and 9, pp. 261-263) are almost adequate to give these results, the missing computation being that which proves that $N(A) < 1043$. This computation has been performed by the present author, using a variation of a method of Titchmarsh (8, pp. 251-252).

THEOREM 19. *For* $2 \leq T$, $|N(T) - F(T)| < R(T)$.

Proof. For $2 \leq T \leq A$, the theorem follows from Theorems 17-18. For $A \leq T$, apply the method of Backlund (10, pp. 354-375) to $(\zeta(s))^N$ instead of to $\zeta(s)$. For instance, the first part of Backlund's discussion (10, pp. 354-361) when applied to $(\zeta(s))^N$ yields

$$|N(T) - F(T)| < \frac{1}{2\pi q N}\int_0^\pi \log|G(re^{i\phi})|\, d\phi$$
$$+ \frac{1}{2} + \frac{4}{\pi T} + \frac{k}{N} - \frac{1}{2qN}\log|G(0)| + \frac{1}{\pi}|S(T)|,$$

where k is a constant independent of N, $r = 1.32$,

$$q = \log \frac{4r}{3} = 0.565314,$$

$$G(s) = \frac{1}{2}\{(\zeta(s + \frac{5}{4} + Ti))^N + (\zeta(s + \frac{5}{4} - Ti))^N\},$$

$$S(T) = \vartheta \log \Gamma\left(\frac{1}{4} + \frac{Ti}{2}\right) - \frac{T}{2} \log \frac{T}{2} + \frac{T}{2} + \frac{\pi}{8}.$$

From Backlund's bounds for $|\zeta(s)|$ (10, pp. 361-369) and the fact that $A \leq T$, one can deduce upper bounds for $|G(re^{i\phi})|$. To deduce an upper bound for $-\log |G(0)|$, note that

$$G(0) = \Re(\zeta(5/4 + iT)^N).$$

Hence we can choose a succession of N's tending to infinity so that

$$\operatorname{Lim} \frac{G(0)}{|\zeta(5/4 + iT)^N|} = 1.$$

However (5, p. 26),

$$|\zeta(5/4 + iT)| \geq \frac{1}{3.425}.$$

All other steps in the proof are strictly analogous to the steps of Backlund's proof.

THEOREM 20. *For* $A \leq \gamma$, $\beta < 1 - \dfrac{1}{17.72 \log \gamma}$.

Proof. Use the procedure outlined by Landau (6, pp. 318-324) using the numerical evaluations of Rosser (5, pp. 30-31) and the fact that

$$18 + 30 \cos \phi + 17 \cos 2\phi + 6 \cos 3\phi + \cos 4\phi$$
$$= 2(1 + \cos \phi)^2 (1 + 2 \cos \phi)^2 \geq 0.$$

Having no further use for the old meaning of $\phi(x)$, we now define

$$\phi(\gamma) = \phi(m, x, \gamma) = \frac{x^{-1/(17.72 \log \gamma)}}{\gamma^{m+1}}.$$

LEMMA 16. *With* K *as in* (9)

$$K \leq x^{-\frac{1}{2}} \sum_{\rho} \frac{1}{|\gamma^{m+1}|} + \sum_{A < \gamma} \phi(\gamma).$$

Proof. By definition,

$$K = \sum_{\beta \leq \frac{1}{2}} \frac{x^{\beta - 1}}{|\gamma^{m+1}|} + \sum_{\beta > \frac{1}{2}} \frac{x^{\beta - 1}}{|\gamma^{m+1}|}.$$

However

$$\sum_{\beta \leq \frac{1}{2}} \frac{x^{\beta - 1}}{|\gamma^{m+1}|} \leq x^{-\frac{1}{2}} \sum_{\beta \leq \frac{1}{2}} \frac{1}{|\gamma^{m+1}|} \leq x^{-\frac{1}{2}} \sum_{\rho} \frac{1}{|\gamma^{m+1}|}.$$

Since $\beta - i\gamma$ is a zero of $\zeta(s)$ if $\beta + i\gamma$ is (4, Theorem 16, p. 48),

$$\sum_{\beta > \frac{1}{2}} \frac{x^{\beta-1}}{|\gamma^{m+1}|} = 2 \sum_{\substack{\beta > \frac{1}{2} \\ \gamma > 0}} \frac{x^{\beta-1}}{\gamma^{m+1}}.$$

However, by Theorem 20,

$$\frac{x^{\beta-1}}{\gamma^{m+1}} < \phi(\gamma).$$

So

$$\sum_{\beta > \frac{1}{2}} \frac{x^{\beta-1}}{|\gamma^{m+1}|} < 2 \sum_{\substack{\beta > \frac{1}{2} \\ \gamma > 0}} \phi(\gamma).$$

By Theorem 18, if $\beta > \frac{1}{2}$ and $\gamma > 0$, then $\gamma > A$. Also $1 - \beta + i\gamma$ is a zero of $\zeta(s)$ if $\beta + i\gamma$ is (4, Theorem 18, p. 48). So

$$2 \sum_{\substack{\beta > \frac{1}{2} \\ \gamma > 0}} \phi(\gamma) \leq \sum_{\gamma > A} \phi(\gamma).$$

LEMMA 17. $\sum_{\rho} \frac{1}{\gamma^2} < 0.0463$; $\sum_{\rho} \frac{1}{|\gamma^3|} < 0.00167$; $\sum_{\rho} \frac{1}{\gamma^4} < 0.0000744$.

Proof. The first result has been proved before (5, pp. 28-30). From it we deduce the second result as follows. Compute

$$\sum_{0 < \gamma \leq 50} \frac{1}{\gamma^2}$$

from Gram's values of γ (11, p. 297). Hence deduce an upper bound for

$$\sum_{50 < \gamma} \frac{1}{\gamma^2}.$$

Dividing by 52.970 (7, p. 59) gives an upper bound for

$$\sum_{50 < \gamma} \frac{1}{\gamma^3}$$

The second result now follows readily. For the third result, note that Gram (11, p. 293) gives the result

$$\tfrac{1}{4} \sum_{\rho} \frac{1}{\alpha^4} = .0000\ 1858\ 6299\ 6426 \cdots,$$

where $\alpha = \gamma + (\beta - \frac{1}{2})i$. Remembering that $\beta = \frac{1}{2}$ if $0 < \gamma \leq A$, we readily deduce the third result.

LEMMA 18. *If* $\log x \leq 942(m+1)$, then

$$\sum_{A < \gamma} \phi(\gamma) < \frac{3.47}{A^{m+1} x^{1/130}} + 0.1592 \int_A^\infty \log \frac{y}{2\pi} \phi(y)\, dy.$$

15

Proof. By a theorem connecting sums with integrals (4, Theorem A, p. 18),

$$\sum_{A<\gamma}\phi(\gamma) = -\int_A^\infty N(y)\phi'(y)\,dy - N(A)\phi(A).$$

However for $A \leq y$,

$$\log x \leq 942(m+1) < 17.72(m+1)\log^2 A \leq 17.72(m+1)\log^2 y,$$

and so $\phi'(y) < 0$. Hence by Theorem 19

$$\sum_{A<\gamma}\phi(\gamma) < -\int_A^\infty (F(y)+R(y))\phi'(y)\,dy - N(A)\phi(A).$$

Integrating by parts gives

$$\sum_{A<\gamma}\phi(\gamma) < \int_A^\infty (F'(y)+R'(y))\phi(y)\,dy + R(A)\phi(A).$$

However

$$R(A)\phi(A) < \frac{3.47}{A^{m+1}x^{1/130}},$$

$$\int_A^\infty F'(y)\phi(y)\,dy = \frac{1}{2\pi}\int_A^\infty \log\frac{y}{2\pi}\,\phi(y)\,dy,$$

and

$$\int_A^\infty R'(y)\phi(y)\,dy = 0.137\int_A^\infty \frac{\phi(y)\,dy}{y} + 0.443\int_A^\infty \frac{\phi(y)\,dy}{y\log y}$$

$$< \frac{0.137}{A\log\dfrac{A}{2\pi}}\int_A^\infty \log\frac{y}{2\pi}\,\phi(y)\,dy + \frac{0.443}{A\log A\log\dfrac{A}{2\pi}}\int_A^\infty \log\frac{y}{2\pi}\,\phi(y)\,dy.$$

LEMMA 19. *If*

$$\log x < \frac{942m^2}{m+0.184},$$

then

$$\int_A^\infty \log\frac{y}{2\pi}\,\phi(y)\,dy < \frac{1+5.454m}{\left(1-\dfrac{m+0.184}{942m^2}\log x\right)m^2A^m x^{1/130}}.$$

Proof. Integrating the left hand side by parts, using

$$\frac{\log\dfrac{y}{2\pi}}{y^{m+1}}$$

as the part to be integrated, we get

$$\int_A^\infty \log\frac{y}{2\pi}\,\phi(y)\,dy = \frac{1+m\log\dfrac{A}{2\pi}}{m^2A^m x^{1/17.72\log A}}$$

$$+\int_A^\infty \frac{\left(m+1/\log\dfrac{y}{2\pi}\right)\log x}{17.72m^2\log^2 y}\log\frac{y}{2\pi}\,\phi(y)\,dy.$$

So

$$\int_A^\infty \log \frac{y}{2\pi}\,\phi(y)\,dy < \frac{1+5.454m}{m^2A^mx^{1/130}} + \frac{m+1/\log \frac{A}{2\pi}\log x}{17.72m^2\log^2 A}\int_A^\infty \log\frac{y}{2\pi}\,\phi(y)\,dy.$$

Solving for $\int_A^\infty \log \frac{y}{2\pi}\,\phi(y)\,dy$ proves the theorem.

THEOREM 21. If

$$\sum_\rho \frac{1}{|\gamma^{m+1}|} \leq k, \qquad \log a < \frac{942m^2}{m+0.184};$$

$$\delta = 2\left\{ a^{-\frac{1}{2}}k + \frac{3.47}{A^{m+1}a^{1/130}} + \frac{0.869m+0.160}{(1-\frac{m+0.184}{942m^2}\log a)\,m^2A^ma^{1/130}} \right\}^{1/(m+1)},$$

$$\epsilon = \frac{\delta}{2}\left\{ \left(\frac{(1+\delta)^{m+1}+1}{2}\right)^m + m \right\},$$

and $1+m\epsilon a < a$, then for $a \leq x$,

$$x(1-\epsilon) - 1.84 < \psi(x) < x(1+\epsilon) - \tfrac{1}{2}\log(1-1/x^2).$$

Proof. Since $K(m,x) \leq K(m,a)$ if $a \leq x$, the theorem follows from Theorem 12, Theorem 16, Lemma 16, Lemma 18, and Lemma 19.

By use of this theorem, Table II was computed. In those cases where $m=4$ or $m=5$, x was large enough so that one could use 0.0000744 (see Lemma 17) as an upper bound for

$$\sum_\rho \frac{1}{|\gamma^5|} \quad\text{and}\quad \sum_\rho \frac{1}{\gamma^6}$$

without appreciably affecting the value of ϵ.

By a method of Rosser (5, pp. 39-40 and Lemma 10, p. 31) one can prove Theorem 22.

THEOREM 22. *If* $\epsilon(x) = (\log x)^{\frac{1}{4}}e^{-\sqrt{(\log x/19)}}$ *and* $e^{4000} \leq x$, *then*

$$(1-\epsilon(x))x < \psi(x) < (1+\epsilon(x))x.$$

Remarks. From (2), Theorem 2, Theorem 7, Table II, Theorem 22, and Gram's table of $\psi(x)$, one can deduce:

For $1 < x \leq e^{100}$ and for $e^{2000} \leq x$,

$$\left(1-\frac{1}{\log x}\right)x < \psi(x) < \left(1+\frac{1}{\log x}\right)x.$$

For $1 < x$, $$\left(1-\frac{2.85}{\log x}\right)x < \psi(x) < \left(1+\frac{2.85}{\log x}\right)x.$$

A curious fact is that $\psi(x)/x$ takes its maximum at $x = 113$, so that one can say that

$$\psi(x) < 1.038821x$$

for all positive x.

In an earlier draft of this paper, a much more complicated, but more accurate, bound for K (depending on an improvement of Theorem 20) was used to deduce the result:

For $1 < x$,

$$\left(1 - \frac{1.8}{\log x}\right)x < \psi(x) < \left(1 + \frac{1.8}{\log x}\right)x.$$

However the computations supporting this result were quite extensive and have never been checked. It is the author's hope to eventually improve the upper bound for K so that one can prove:

For $e^{80} < x$,

$$\left(1 - \frac{0.95}{\log x}\right)x < \psi(x) < \left(1 + \frac{0.95}{\log x}\right)x.$$

If this could be done, one could replace the upper bounds on x and n in Theorems 23-28 by infinity.

The range $1{,}000{,}000 \leq x$. By (2) and Table II we may infer that:

(10) For $e^{13.8} \leq x$, $(1 - 0.0393)x < \theta(x) < (1 + 0.0376)x$.

(11) For $e^{15} \leq x$, $(1 - 0.0328)x < \theta(x) < (1 + 0.0321)x$.

(12) For $e^{20} \leq x$, $(1 - 0.02)x < \theta(x) < (1 + 0.0199)x$.

For $e^{30} \leq x$, we may take Table II and Theorem 22 as referring to $\theta(x)$ as well as to $\psi(x)$. Hence, with the help of Theorem 2, Corollary and Theorem 7, Corollary one can readily deduce Theorems 23, 24, 29C, and 30C. Similarly, one can deduce Lemmas 20-21.

LEMMA 20. *For* $2 \leq x$,

$$\left(1 - \frac{2.85}{\log x}\right)x < \theta(x) < \left(1 + \frac{2.85}{\log x}\right)x.$$

LEMMA 21. *For* $e^{2000} \leq x$,

$$\left(1 - \frac{0.96}{\log x}\right)x < \theta(x) < \left(1 + \frac{0.96}{\log x}\right)x.$$

Remarks. Because of Theorem 2 and (10), we may infer that $\theta(x) < 1.0376x$ for all positive x. Hence

$$\prod_{p \leq x} p < (2.83)^x$$

for positive x. Because of Theorem 5, Theorem 7, and (10), we may infer that $0.9607x < \theta(x)$ for $2600 \leq x$. Computation and Theorem 6 gives the result that $0.6932x < \theta(x)$ for $29 \leq x$. Hence

$$(2.61)^x < \prod_{p \leq x} p \text{ for } 2600 \leq x.$$

$$2^x < \prod_{p \leq x} p \text{ for } 29 \leq x.$$

In order to derive Theorems 25, 26, 29A, and 30A, we proceed as follows. First note that by (7) and Theorem 24,

$$\pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \int_2^x \frac{dy}{\log^2 y} + \int_2^x \frac{dy}{\log^3 y} \tag{13}$$

for $2 \leq x \leq e^{100}$. Since

$$\int \frac{dy}{\log^3 y} = -\frac{y}{2\log^2 y} + \int \frac{dy}{2\log^2 y}$$

and

$$\int \frac{dy}{\log^2 y} = -\frac{y}{\log y} + li(y),$$

one can compute the value of the right side of (13) for any x. In particular one can show that for $e^{13.8} \leq x$, the right side of (13) is less than $x/(\log x - 2)$. Theorem 25 now follows by use of Theorem 9.

By (7),

$$\pi(x) > \frac{\theta(x)}{\log x} + \int_{41}^x \frac{\theta(y)dy}{y\log^2 y}$$

if $41 \leq x$. So for $41 \leq x \leq e^{100}$,

$$\pi(x) > \frac{x}{\log x} - \frac{x}{\log^2 x} + \int_{41}^x \frac{dy}{\log^2 y} - \int_{41}^x \frac{dy}{\log^3 y}$$

by Theorem 23. For $e^{13.8} \leq x$, the right side of this is greater than $x/\log x$. So Theorem 26 follows by use of Theorem 8.

Now assume $e^{100} \leq x$. Then by (7) and Lemma 20

$$\pi(x) < \frac{x}{\log x} + \frac{2.85x}{\log^2 x} + \int_2^x \frac{dy}{\log^2 y} + 2.85\int_2^x \frac{dy}{\log^3 y}. \tag{14}$$

However

$$\int_2^x \frac{dy}{\log^2 y} = \frac{x}{\log^2 x} - \frac{2}{\log^2 2} + 2\int_2^x \frac{dy}{\log^3 y} < \frac{x}{\log^2 x} + 2\int_2^x \frac{dy}{\log^3 y},$$

$$\int_2^x \frac{dy}{\log^3 y} < \frac{x}{\log^3 x} + 3\int_2^x \frac{dy}{\log^4 y},$$

and

$$\int_2^x \frac{dy}{\log^4 y} = \int_2^{x^{\frac{1}{2}}} \frac{dy}{\log^4 y} + \int_{x^{\frac{1}{2}}}^x \frac{dy}{\log^4 y} < x^{\frac{1}{2}} + \frac{x}{\log^4 (x^{\frac{1}{2}})}.$$

From these inequalities, one can easily show that the right side of (14) is less than $x/(\log x - 4)$, and so prove one half of Theorem 29A. The proof of the other half is similar.

Now assume $e^{2000} \leq x$. Then by (7) and Lemmas 20-21,

$$\pi(x) < \frac{x}{\log x} + \frac{0.96x}{\log^2 x} + \int_2^x \frac{dy}{\log^2 y} + 2.85 \int_2^x \frac{dy}{\log^3 y}.$$

This is treated in the same manner that (14) was, and one half of Theorem 30A is proved thereby. The other half is proved in a similar manner.

We now consider Theorem 27, Theorem 29B, and Theorem 30B. To prove the last two, we will need to prove the results

$$(n+1)\log(n+1) + (n+1)\log\log(n+1) - (n+1) - A(n+1) < p(n)$$

for $A = 3$ and $A = 1$ respectively. Let $a \leq p(n)$, and suppose that for $a \leq x$, $\theta(x) < (1+\epsilon)x$. Let $80{,}000 \leq n$. Now suppose that

$$p(n) \leq (n+1)\log(n+1) + (n+1)\log\log(n+1) - 2(n+1).$$

Since $80{,}000 \leq n$, we can infer

$$p(n) < n\log n + n\log\log n - 2n + 2\log n.$$

Then by Lemma 9,

$$n\log n + n\log\log n - n - li(n) < (1+\epsilon)(n\log n + n\log\log n - 2n + 2\log n).$$

That is

$$(15) \qquad 1 + 2\epsilon < \frac{li(n)}{n} + \frac{(2+2\epsilon)\log n}{n} + \epsilon(\log n + \log\log n).$$

Now if $a = e^{13.8}$ and $a \leq p(n)$, then $82{,}395 \leq n$. Also, by (10), we can take $\epsilon = 0.0376$. Then (15) is false for $n \leq e^{20}$. If $a = e^{20}$, we can take $\epsilon = 0.0199$ by (12). Then (15) is false for $n \leq e^{40}$. If $a = e^{40}$, we can take $\epsilon = 0.0119$ by Table II. Then (15) is false for $n \leq e^{80}$. And so on up to $n \leq e^{100}$. This with Theorem 10 proves Theorem 27, and a similar procedure proves one half of Theorem 29B and Theorem 30B.

We now consider Theorem 28. Suppose $a \leq p(n)$, and that for $a \leq x$, $(1-\epsilon)x < \theta(x)$. Also suppose that $p(r) < r\log r + r\log\log r$ for $16 \leq r < n$ and that $n\log n + n\log\log n \leq p(n)$. Then by Lemma 10,

$$(1-\epsilon)(n\log n + n\log\log n) < n\log n + n\log\log n - n + \frac{n\log\log n}{\log n}.$$

That is,

$$1 < \frac{\log\log n}{\log n} + \epsilon(\log n + \log\log n).$$

For $83{,}000 \leq n \leq e^{95}$, this is false, and so Theorem 28 follows.

Lemma 22. *If* $e^{95} \leq n$, *then*

$$\theta(p(n)) < n\log n + n\log\log n - n + \frac{2n\log\log n}{\log n}.$$

Proof. Since $p(n) < n\log n + 2n\log\log n$ (5, Theorem 2, p. 40),

$$\theta(p(n)) < \theta(p(15)) + \sum_{r=16}^{n-1}\log(r\log r + 2r\log\log r)$$
$$+ \log(n\log n + 2n\log\log n).$$

From here, proceed as in the proof of Lemma 10.

By use of Lemma 22, we can readily prove the rest of Theorem 29B and Theorem 30B.

Table I.

x	$\theta(x)$	x	$\theta(x)$
2000	1939.839 200	6500	6408.907 367
2500	2433.602 748	7000	6920.421 031
3000	2932.359 205	7500	7364.857 418
3500	3409.457 181	8000	7875.150 386
4000	3911.145 393	8500	8343.999 665
4500	4412.188 301	9000	8870.374 997
5000	4911.695 346	9500	9418.368 776
5500	5391.372 236	10000	9895.991 380
6000	5893.297 458		

Table II.

Against values of b are tabulated those values of ϵ such that one can deduce from Theorem 21 that $(1-\epsilon)x < \psi(x) < (1+\epsilon)x$ for $e^b \leq x$. Since the value of ϵ that can be deduced from Theorem 21 depends on a preassigned value of m, the values of m which were used are tabulated with b and ϵ.

b	m	ϵ	b	m	ϵ
13.8	2	0.0381	500	3	0.00511
15	2	0.0321	550	3	0.00467
20	2	0.0199	600	3	0.00427
30	2	0.0179	650	3	0.00392
40	3	0.0119	700	3	0.00356
60	4	0.0101	800	2	0.00294
80	4	0.00983	900	2	0.00235
90	5	0.00938	1000	2	0.00194
100	5	0.00932	1300	2	0.00104
300	4	0.00710	2000	3	0.000383
400	4	0.00609	2300	3	0.000255
450	4	0.00567	3000	4	0.000158

PRINCETON UNIVERSITY.

BIBLIOGRAPHY.

1. J. P. Gram, "Maengden af Primtal under en given Graense," *K. Danske Vidensk. Selskabs Skrifter*, ser. 6, vol. 2 (1881-86), pp. 183-308.

2. G. W. Jones, *Logarithmic Tables*, Macmillan and Co., Seventh Edition, 1898.

3. D. N. Lehmer, *List of Prime Numbers from* 1 *to* 10,006,721, Carnegie Institution of Washington Publication No. 165, 1914.

4. A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge Tract No. 30, 1932, London.

5. J. B. Rosser, "The n-th prime is greater than $n \log n$," *Proceedings of the London Mathematical Society*, ser. 2, vol. 45 (1939), pp. 21-44.

6. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen.*

7. J. I. Hutchinson, "On the roots of the Riemann zeta function," *Transactions of the American Mathematical Society*, vol. 27 (1925), pp. 49-60.

8. E. C. Titchmarsh, "The zeros of the Riemann zeta-function," *Proceedings of the Royal Society of London*, Series A, vol. 151 (1935), pp. 234-255.

9. E. C. Titchmarsh, "The zeros of the Riemann zeta-function," *Proceedings of the Royal Society of London*, Series A, vol. 157 (1936), pp. 261-263.

10. R. J. Backlund, "Über die Nullstellen der Riemannschen Zetafunktion," *Acta Mathematica*, vol. 41 (1917), pp. 345-375.

11. J. P. Gram, "Note Sur les Zéros de la Fonction $\zeta(s)$ de Riemann," *Acta Mathematica*, vol. 27 (1903), pp. 289-304.